A Converse Theorem without Root Numbers

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What is a Converse Theorem?

“A converse theorem characterizes automorphic forms in terms of analytic properties of their $L$-functions.”
A classical result

Let $f : H \to \mathbb{C}$ have a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

Can associate to $f$ the completed $L$-function

$$\Lambda(s; f) = (2\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s}$$
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Theorem (Hecke ’36)

\( f \) is a modular form for \( SL_2(\mathbb{Z}) \) of weight \( k \) if and only if \( \Lambda(s; f) \)

(i) has an analytic continuation to the whole \( s \)-plane

(ii) is bounded in vertical strips

(iii) satisfies the functional equation

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\Lambda(s; f) = (-1)^{k/2}\Lambda(k - s; f)
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The if part of this statement is a prototypical example of a Converse theorem.
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Weil (1967) proved a converse theorem requiring a family of ‘twisted’ $L$-functions.
Weil’s setup

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- Associate to them a pair of functions $f$, $\tilde{f}$

$$f(z) = \sum_{n=1}^{\infty} \lambda_n n^{\frac{k-1}{2}} e^{2\pi i nz} \quad \text{and} \quad \tilde{f}(z) = \sum_{n=1}^{\infty} \tilde{\lambda}_n n^{\frac{k-1}{2}} e^{2\pi i nz}. $$
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\]

- Define the \( L \)-function twisted by the Dirichlet character \( \chi \)

\[
\Lambda(s; \lambda, \chi) := \Gamma_{\mathbb{C}} \left( s + \frac{k-1}{2} \right) \sum_{n=1}^{\infty} \lambda_n \chi(n) n^{-s}.
\]
Weil’s Converse theorem

Weil showed that if the $L$-functions defined above are ‘nice’ for every Dirichlet character $\chi$ with conductor $q$ relatively prime to $N$ and satisfy the functional equation

$$\Lambda(s; \lambda, \chi) = C_\chi(q^2N)^{\frac{1}{2} - s} \Lambda(1 - s; \tilde{\lambda}, \bar{\chi}),$$

then $f$ is a modular form of level $N$ and weight $k$. 

The complex number $C_\chi = i^k \xi(q) \chi(-N)/\tau(\bar{\chi})$, with $\tau(\chi)$ the Gauss sum for $\chi$ and $\xi$ the nebentypus character of $f$, is called the root number of the functional equation.
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Assume

- the central character $\chi$ of $\pi$ is an idele class character, and
- the $L$-function $L(s; \pi) = \prod_v L(s; \pi_v)$ converges in some right half plane.
Theorem (Jacquet and Langlands ’70)

Suppose, for each idele class character $\omega$, the twisted $L$-functions $L(s; \pi \otimes \omega)$ and $L(s; \bar{\pi} \otimes \omega^{-1})$ can be continued to entire functions of $s$, are bounded in vertical strips and satisfy the functional equation

$$L(s; \pi \otimes \omega) = \varepsilon(s; \pi \otimes \omega)L(1 - s; \bar{\pi} \otimes \omega^{-1}).$$

Then $\pi$ is a cuspidal automorphic representation.
For each $\xi = \bigotimes_v \xi_v \in V_\pi$ let $W_\xi = \prod_v W_{\xi_v} \in \mathcal{W}(\pi, \psi)$ and set

$$\varphi_\xi(g) = \sum_{\gamma \in k^\times} W_\xi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right).$$

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Show, for all $g$

$$\varphi_\xi(wg) = \varphi_\xi(g),$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This shows $\varphi_\xi$, and hence $\pi$, is automorphic.
Can we relax the requirement of precise $\varepsilon$-factor?

**Theorem (Booker '19)**

Let $\pi$ be an irreducible admissible representation of $GL_2(\mathbb{A}_\mathbb{Q})$ with automorphic central character and conductor $N$. Suppose each $\pi_v$ is unitary and that $\pi_\infty$ is a discrete series or limit of discrete series representation. For each unitary idele class character $\omega$ of conductor $q$ coprime to $N$, suppose the completed $L$-functions $\Lambda(s, \pi \otimes \omega)$ and $\Lambda(s, \tilde{\pi} \otimes \omega^{-1})$ continue to entire functions on $\mathbb{C}$, are bounded in vertical strips and satisfy a functional equation of the form

$$\Lambda(s, \pi \otimes \omega) = \varepsilon_{\omega} \left( \frac{Nq^2}{2} \right)^{1/2 - s} \Lambda(1 - s, \tilde{\pi} \otimes \omega^{-1})$$

for some complex number $\varepsilon_{\omega}$. Then there is a cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ such that $\Pi_\infty \sim \pi_\infty$ and $\Pi_v \sim \pi_v$ at every finite $v$ at which $\pi_v$ is unramified.
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- I prove a version for a rational function field
- The values for $\epsilon_\omega$ require some additional (natural) constraints
The case of a rational function field

- $F = \mathbb{F}_q(t)$
- $\mathbb{A}$ the adele ring of $F$
- Fix a place $\infty$ of $F$
- $\pi$ an irreducible admissible generic representation of $GL_2(\mathbb{A})$ with conductor $\alpha$, and automorphic central character $\chi$
The case of a rational function field

**Theorem (A)**

For each unitary idele class character $\omega$ whose conductor $\mathfrak{f}$ is disjoint from $\alpha$, assume the $L$-function $L(s, \pi \otimes \omega)$ continues to a holomorphic function on $\mathbb{C}$ and satisfies the functional equation

$$L(s, \pi \otimes \omega) = \epsilon_\omega |af^2|^{s-\frac{1}{2}} L(1 - s, \tilde{\pi} \otimes \omega^{-1}),$$

where the complex number $\epsilon_\omega$ is such that

(i) if $\omega$ is unramified or ramified only at $\infty$, then $\epsilon_\omega = 1$, and

(ii) for any unramified unitary idele class character $\omega'$, we have $\epsilon_{\omega'} \omega = \epsilon_\omega$.

Then there is a cuspidal automorphic representation $\Pi$ so that $\Pi_v \cong \pi_v$ at all places $v$ away from the support of the divisor $\alpha$. 
Key ingredients in the proof

- Basic idea of showing \( \varphi_\xi(wg) = \varphi_\xi(g) \) remains the same
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- Average the subsequent equality we get for the twisted $\varphi$ and its dual over all unitary characters \textit{mod a fixed divisor}
Key ingredients in the proof

- Basic idea of showing $\phi_\xi(wg) = \phi_\xi(g)$ remains the same
- Define a notion of twist of $\phi$ by a character $\text{mod a divisor}$
- Derive a functional equation for the Dirichlet series associated to these twisted variants of $\phi$
- Average the subsequent equality we get for the twisted $\phi$ and its dual over all unitary characters $\text{mod a fixed divisor}$
- Primes in arithmetic progression in a rational function field
Twists mod a conductor

Let \( \xi^0 = \bigotimes_v \xi^0_v \in V_\pi \), where \( \xi^0_v \) is the new vector in \( V_{\pi_v} \). Like before, set

\[
\varphi_{\xi^0}(g) = \sum_{\gamma \in k^\times} W_{\xi^0} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right).
\]

For \( \omega \) an idele class character, define

\[
I(s; \varphi_{\xi^0}, \omega) = \int_{\mathbb{A}^\times} W_{\xi^0} \left( \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) \omega(u) |u|^{s-\frac{1}{2}} \, du.
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$$\varphi_{\xi^0}(g) = \sum_{\gamma \in k^\times} W_{\xi^0} \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right).$$

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$$I(s; \varphi_{\xi^0}, \omega) = \int_{\mathbb{A}^\times} W_{\xi^0} \left( \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) \omega(u) |u|^{s - \frac{1}{2}} \, du.$$

- If $\omega$ is ramified at any place $\pi$ is unramified, this integral becomes zero.
To still be able to work with an explicit function in the integral representation and get something non-zero, we define a variant of \( \varphi = \varphi \xi_0 \).
Twists mod a conductor

To still be able to work with an explicit function in the integral representation and get something non-zero, we define a variant of $\varphi = \varphi_{\xi_0}$. Let $f_0$ be a divisor and $\tau$ an idele class character with conductor dividing $f_0$. Denote by $\varphi(x, y)$ the value $\varphi \left( \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \right)$. Denote by $\varphi(x, y)$ the value $\varphi \left( \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \right)$.
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\[
\varphi_{\tau,f_0}(x, y) = \int_{\prod_v \mathcal{O}^\times_v} \tau(u) \varphi \left( \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & wu \\ 0 & 1 \end{pmatrix} \right) du,
\]

where \( w \) is an adele given in terms of \( f_0 \).
Twists mod a conductor

Working with the integral $I(s; \varphi_{\omega, f_0}, \omega)$ instead, we can pick out local $L$-factors of $L(s, \pi \otimes \omega)$ even at places where $\omega$ is ramified. By varying $f_0$, we get finer control on what terms in the Dirichlet series corresponding to $L(s, \pi \otimes \omega)$ we pick up.
We can explore the role of root numbers in functional equations in the context converse theorems. The Langlands-Shahidi method gives a well developed theory of $\varepsilon$-factors, so I don’t see any direct application. However, if we had a method of constructing $L$-functions that did not give precise $\varepsilon$-factors, converse theorems not requiring root numbers could be useful.
Andrew Booker (2019)
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Hervé Jacquet and Robert Langlands (1970)
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Thank You!