Diameters of compact arithmetic hyperbolic surfaces

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For the purpose of this talk, a hyperbolic surface will be \( \Gamma \backslash \mathbb{H} \) for a discrete subgroup \( \Gamma \subset SL_2(\mathbb{R}) \), where the action is given by Möbius transformations \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \).
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Basic examples for $\Gamma$ include:

- $\{\pm I\}$,
- $\text{SL}_2(\mathbb{Z})$,
- $\Gamma_0(N) = \{ (a \ b) \in \text{SL}_2(\mathbb{Z}) | c \equiv 0 \text{ mod}(N) \}$,
- $\Gamma(N) = \{ \gamma \in \text{SL}_2(\mathbb{Z}) | \gamma \equiv I \text{ mod}(N) \}$.
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- $\Gamma(N) = \{ \gamma \in \text{SL}_2(\mathbb{Z}) | \gamma \equiv I \text{ mod}(N) \}$.

Except for the first, these are all examples of arithmetic lattices in $\text{SL}_2(\mathbb{R})$. However, they are not co-compact.
We start with a quaternion algebra $B = \left( \frac{a,b}{\mathbb{Q}} \right)$, $a, b \in \mathbb{Q}^\times$, which we assume to be split (indefinite) over the reals, i.e. $B \otimes \mathbb{R} \cong \text{Mat}_{2 \times 2}(\mathbb{R}) \Leftrightarrow a > 0$ or $b > 0$.

Recall $\left( \frac{a,b}{\mathbb{Q}} \right)$ is the $\mathbb{Q}$-algebra generated by $1, i, j, k$ with the relations

$$i^2 = a, \quad j^2 = b, \quad k = ij = -ji.$$
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Next, we introduce an integral structure $R \subset B$, an order, that is a lattice which is also closed under multiplication. For us, $R$ will either be maximal or an intersection of two maximal orders, a so-called Eichler order.
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$$R \subset B \otimes \mathbb{R} \cong \text{Mat}_{2\times 2}(\mathbb{R})$$

$$\Gamma := R^1 \subset (B \otimes \mathbb{R})^1 \cong \text{SL}_2(\mathbb{R})$$
$B$ an indefinite quaternion algebra over $\mathbb{Q}$, $R \subset B$ an Eichler order, $\Gamma = R^1$ the set of proper units.
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- $B = (\frac{1,1}{\mathbb{Q}}) \simeq \text{Mat}_{2 \times 2}(\mathbb{Q})$, $R = \text{Mat}_{2 \times 2}(\mathbb{Z})$ (maximal), $\Gamma = \text{SL}_2(\mathbb{Z})$,
- $B = (\frac{7,5}{\mathbb{Q}})$, $R = \langle 1, \frac{1+j}{2}, i, \frac{1+i+j+k}{2} \rangle_\mathbb{Z}$ (maximal), $\Gamma = R^1$,
- $B = (\frac{77,-1}{\mathbb{Q}})$, $R = \langle 1, \frac{1+i}{2}, j, \frac{j+k}{2} \rangle_\mathbb{Z}$ (maximal), $\Gamma = R^1$. 

R. S. Steiner

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The discriminant $\mathfrak{D}$ of $B$ is the product of the (finite) primes $p$ for which $B \otimes \mathbb{Q}_p \ncong \text{Mat}_{2 \times 2}(\mathbb{Q}_p)$. 

R. S. Steiner  
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The volume of $\Gamma \backslash \mathbb{H}$ is $V = (\mathfrak{D}\mathfrak{N})^{1+o(1)}$ and $\Gamma$ is co-compact iff $\mathfrak{D} > 1$. 

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Ford fundamental domains of the previous co-compact arithmetic lattices after a Cayley transformation $\mathbb{H} \to \mathcal{D}$.

Figure 13: $F = \mathbb{Q}$, $D = 35$, $\mu(U) = 25.1327412287$.

Figure 31: $F = \mathbb{Q}$, $D = 77$, $\mu(U) = 62.8318530718$.

These images are a courtesy of James Rickards.
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Prior and new results

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**Theorem (S.)**

Let $\Gamma$ be an arithmetic co-compact lattice stemming from an Eichler order of square-free level in an indefinite quaternion algebra over $\mathbb{Q}$. Then, for every point $w$ on the hyperbolic surface $\Gamma \backslash \mathbb{H}$ of volume $V$, almost every point $z \in \Gamma \backslash \mathbb{H}$ satisfies

$$\min_{\gamma \in \Gamma} d(\gamma z, w) \leq (1 + o(1)) \log(V).$$
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Then, it satisfies to show for $T_0 = (1 + \epsilon) \log(V)$, that

\[
\nu_{prob}(w \in \Gamma \setminus \mathbb{H} \mid A_{T_0} B_z(w) = 0) \ll V^2 \|A_{T_0} B_z - \langle B_z, 1 \rangle 1\|_2^2 = o(1).
\]
Proof sketch

\[ V^2 \| A_{T_0} B_z - \langle B_z, 1 \rangle 1 \|_2^2 \]
\[ \ll T_0^2 \sum_{0 < \lambda_j \leq \frac{1}{4}} (e^{-\frac{T_0}{2}})^{2(1 - \sqrt{1 - 4\lambda_j})} |u_j(z)|^2 + T_0^2 e^{-T_0} V^2 \| B_z \|_2^2. \]
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Use \( \| B_z \|_2^2 \ll V^{-1} \), Cauchy–Schwarz to split off the exceptional Maaß forms \( u_j \), a strong density estimate for one of the factors, and a sharp estimate on the fourth moment of exceptional Maaß form by Khayutin–Nelson–S. (soon to appear) for the other factor.
Thank you for listening!