Diameters of compact arithmetic hyperbolic surfaces

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Compact arithmetic hyperbolic surfaces

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Basic examples for Γ include:

- $\{\pm I\}$,
- $\mathrm{SL}_2(\mathbb{Z})$,
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) | c \equiv 0 \bmod (N)\}$,
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Except for the first, these are all examples of *arithmetic lattices* in $\mathrm{SL}_2(\mathbb{R})$. However, they are not co-compact.

Compact arithmetic hyperbolic surfaces

We start with a quaternion algebra $B = \left(\frac{a,b}{\mathbb{Q}}\right)$, $a, b \in \mathbb{Q}^\times$, which we assume to be split (indefinite) over the reals, i.e. $B \otimes \mathbb{R} \cong \mathrm{Mat}_{2\times 2}(\mathbb{R})$ ($\Leftrightarrow a > 0$ or $b > 0$).

Recall $\left(\frac{a,b}{\mathbb{Q}}\right)$ is the $\mathbb{Q}$-algebra generated by $1, i, j, k$ with the relations

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$$R \subset B \otimes \mathbb{R} \cong \mathrm{Mat}_{2\times 2}(\mathbb{R})$$
$$\Gamma := R^1 \subset (B \otimes \mathbb{R})^1 \cong \mathrm{SL}_2(\mathbb{R})$$

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- $B = \left(\frac{7,5}{\mathbb{Q}}\right)$, $R = \langle 1, \frac{1+j}{2}, i, \frac{1+i+j+k}{2} \rangle_{\mathbb{Z}}$ (maximal), $\Gamma = R^1$,
- $B = \left(\frac{77,-1}{\mathbb{Q}}\right)$, $R = \langle 1, \frac{1+i}{2}, j, \frac{j+k}{2} \rangle_{\mathbb{Z}}$ (maximal), $\Gamma = R^1$.

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The volume of $\Gamma\backslash\mathbb{H}$ is $V = (\mathfrak{D}\mathfrak{N})^{1+o(1)}$ and Γ is co-compact iff $\mathfrak{D} > 1$.

Ford fundamental domains of the previous co-compact arithmetic lattices after a Cayley transformation $\mathbb{H} \to \mathcal{D}$.

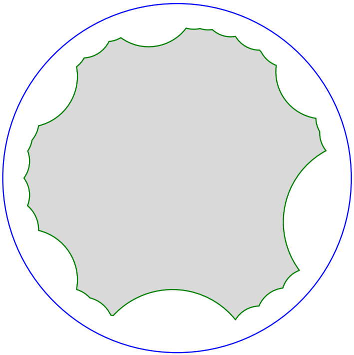

Figure 13: $F = \mathbb{Q}$, $\mathfrak{D} = 35$, $\mu(U) = 25.1327412287$.

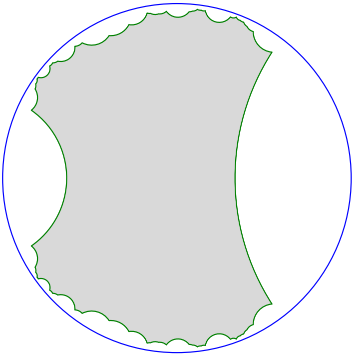

Figure 31: $F = \mathbb{Q}$, $\mathfrak{D} = 77$, $\mu(U) = 62.8318530718$.

These images are a courtesy of James Rickards.

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- Bounding the size of generators of Γ,
- Giving a runtime complexity for computing these domains, generators, reduced word problem w.r.t. these generators, computing intersection numbers of geodesics,... (work by Voight, Rickards, etc.),
- Indefinite analogue of the LPS-graphs, a type of Ramanujan graphs which admit small diameter due to the large spectral gap. (work by Lubotzky, Phillips, Sarnak, Golubev, Kamber, etc.)

Prior and new results

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Theorem (S.)

Let Γ be an arithmetic co-compact lattice stemming from an Eichler order of square-free level in an indefinite quaternion algebra over $\mathbb{Q}$. Then, for every point w on the hyperbolic surface $\Gamma\backslash\mathbb{H}$ of volume V, almost every point $z \in \Gamma\backslash\mathbb{H}$ satisfies

$$\min_{\gamma\in\Gamma} d(\gamma z, w) \leq (1 + o(1)) \log(V).$$

Proof sketch

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Let A_T be some operator that dissipates the mass at unit speed evaluated at time T. (Think geodesic flow projected down to the surface.)
Then, it satisfies to show for $T_0 = (1+\epsilon)\log(V)$, that

$$\nu_{prob}(w \in \Gamma\backslash\mathbb{H} \,|\, A_{T_0}B_z(w) = 0) \ll V^2\|A_{T_0}B_z - \langle B_z, 1\rangle 1\|_2^2 = o(1).$$

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$$V^2\|A_{T_0}B_z - \langle B_z, 1\rangle 1\|_2^2$$
$$\ll T_0^2 \sum_{0<\lambda_j\le\frac{1}{4}} (e^{-\frac{T_0}{2}})^{2(1-\sqrt{1-4\lambda_j})}|u_j(z)|^2 + T_0^2 e^{-T_0} V^2\|B_z\|_2^2.$$

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Use $\|B_z\|_2^2 \ll V^{-1}$, Cauchy–Schwarz to split off the exceptional Maaß forms u_j, a strong density estimate for one of the factors, and a sharp estimate on the fourth moment of exceptional Maaß form by Khayutin–Nelson–S. (soon to appear) for the other factor.

Thank you for listening!