QUADRATIC FORMS REPRESENTING ALL ODD POSITIVE INTEGERS

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Abstract. We consider the problem of classifying all positive-definite integer-valued quadratic forms that represent all positive odd integers. Kaplansky considered this problem for ternary forms, giving a list of 23 candidates, and proving that 19 of those represent all positive odds. (Jagy later dealt with a 20th candidate.) Assuming that the remaining three forms represent all positive odds, we prove that an arbitrary, positive-definite quadratic form represents all positive odds if and only if it represents the odd numbers from 1 up to 451. This result is analogous to Bhargava and Hanke's celebrated 290-theorem. In addition, we prove that these three remaining ternaries represent all positive odd integers, assuming the Generalized Riemann Hypothesis.

This result is made possible by a new analytic method for bounding the cusp constants of integer-valued quaternary quadratic forms Q with fundamental discriminant. This method is based on the analytic properties of Rankin-Selberg L-functions, and we use it to prove that if Q is a quaternary form with fundamental discriminant, the largest locally represented integer n for which $Q(\vec{x}) = n$ has no integer solutions is $O(D^{2+\epsilon})$.