THE EXPLICIT SATO-TATE CONJECTURE AND DENSITIES PERTAINING TO LEHMER-TYPE QUESTIONS

JESSE THORNER

Abstract. Let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_k^{\text{new}}(\Gamma_0(N))$ be a normalized Hecke eigenform with N squarefree. For a prime p, define $\theta_p \in [0, \pi]$ to be the angle for which $a(p) = 2p^{(k-1)/2}\cos(\theta_p)$. Let $I = [\alpha, \beta] \subset [0, \pi]$, and let $\mu_{ST}(I) = \int_\alpha^\beta \frac{2}{\pi}\sin^2(\theta)\, d\theta$ be the Sato-Tate measure. We prove, assuming that the symmetric power L-functions of f are automorphic and satisfy the Generalized Riemann Hypothesis, that

$$|\#\{p \in [x, 2x] : \theta_p \in I\} - (\pi(2x) - \pi(x))\mu_{ST}(I)| = O\left(\frac{x^{3/4}\log(Nkx)}{\log(x)}\right),$$

where the implied constant is $\frac{2\sqrt{15}}{3}$. This bound decreases by a factor of $\sqrt{\log(x)}$ if we let $I = [\frac{\pi}{2} - \frac{1}{2}\Delta, \frac{\pi}{2} + \frac{1}{2}\Delta]$, where Δ is small. This allows us to compute lower bounds for the density of positive integers n for which $a(n) \neq 0$. In particular, we prove that if τ is the Ramanujan tau function, then

$$\lim_{x\to\infty} \frac{\#\{n \in [1, x] : \tau(n) \neq 0\}}{x} > 1 - 8.8 \cdot 10^{-6}.$$