Probabilistic methods and modular forms

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PLAN OF TALK

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- Part-I: Introduction to theory of equidistribution.

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- Part-II: Distribution of gaps between elements of equidistributed sequences.

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- Part-II: Distribution of gaps between elements of equidistributed sequences.
- Part-III: Sato-Tate conjecture and beyond.

Part-I: Introduction to theory of equidistribution.

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(P.G.L. Dirichlet, 1805 – 1859)
(Source: wikipedia.org)

1842 - Dirichlet showed that there are infinitely many elements of this sequence in any neighborhood of 0.

(Leopold Kronecker, 1823 – 1891)
(Source: wikipedia.org)

1884 - Kronecker showed that this sequence is in fact dense throughout the interval [0, 1].

(P. Bohl)
(1865-1921)

(H. Weyl)
(1885-1955)

(W. Sierpinski)
(1882-1969)

(Source: wikipedia.org)

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1909 - Piers Bohl.
1910 - Herman Weyl and Waclaw Sierpinski, investigated the following question:
Question How the sequence $\{n\theta\}$ is distributed in the unit circle, when θ is irrational?

General principles of equidistribution

Let $\{x_n\}$ be a sequence of real numbers in the unit interval $[0, 1]$. For a subset I of $[0, 1]$, and for a fixed natural number N, let

$$A_I(N) := \#\{1 \leq n \leq N; \, x_n \in I\}.$$

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Definition 1

$\{x_n\}$ *is said to be* **equidistributed** *in the unit interval if for any* $I = [a, b] \subset [0, 1]$, *we have*

$$\lim_{N \to \infty} \frac{A_I(N)}{N} = b - a.$$

Definition 2

A sequence of real numbers $\{x_n\}$ *is said to be* **equidistributed** mod 1 *if the sequence* $\{x_n\}$ *is equidistributed in* $[0, 1]$.

A sequence $\{x_n\}$ of real numbers is said to be **equidistributed** mod 1 $\Leftrightarrow$ for every $I \subset [0,1]$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_I(x_n) = \int_0^1 \chi_I(x)dx.$$

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$\Leftrightarrow$ for all (complex valued) Riemann integrable functions $f(x)$ of period 1,

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$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx.$$

$\Rightarrow$ For all non-zero $m \in \mathbb{Z}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m x_n} = 0.$$

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$\{n^2\theta\}$ *is equidistributed in the unit interval.*

In that paper he gave a criterion for equidistribution in terms exponential sum.

Weyl's Criterion

Theorem 4

Weyl's criterion: *A sequence* $\{x_n\}$ *is e.d* $\pmod 1$ *if and only if*

$$c_m := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(mx_n) = 0$$

for every $m \in \mathbb{Z},\ m \neq 0,\ e(t) = e^{2\pi i t}$.

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Application: If $\theta \notin \mathbb{Q}$, then $\{n\theta\}$ e.d $\pmod 1$ in $[0, 1]$.

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Application: If $\theta \notin \mathbb{Q}$, then $\{n\theta\}$ e.d (mod 1) in $[0,1]$. If $m \neq 0$,

$$\begin{aligned} &\frac{1}{N} \left| \sum_{n=1}^{N} e(mn\theta) \right| \\ &= \frac{1}{N} \left| \frac{\sin(\pi mN\theta)}{\sin(\pi m\theta)} \right| \\ &\to 0 \text{ as } N \to \infty. \end{aligned}$$

Definition 5

Consider finite multi sets A_n with $\#A_n \to \infty$ as $n \to \infty$.
$\{A_n\}$ is **set-equidistributed** *with respect to a probability measure μ if for every $[a,b] \subset [0,1]$,*

$$\lim_{n\to\infty} \frac{\#\{t \in A_n : \ t \in [a,b]\}}{\#A_n} = \int_a^b d\mu.$$

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In particular, if

$$A_n = \{x_1, x_2, ..., x_n\}$$

then the definition is the definition of an equidistributed sequence with respect to μ.

The definition of equidistribution can be generalised to arbitrary interval in the following ways

Definition 6

Consider finite multi sets A_n in an interval $[\alpha, \beta]$with $\#A_n \to \infty$ as $n \to \infty$. $\{A_n\}$ is **set-equidistributed** *in $[\alpha, \beta]$ with respect to a probability measure μ if for every continuous function $f : [\alpha, \beta] \to \mathbb{C}$, the following limit holds:*

$$\lim_{n\to\infty} \frac{1}{\#A_n} \sum_{t\in A_n} f(t) = \frac{1}{\beta - \alpha} \int_\alpha^\beta d\mu.$$

In this talk, we will be interested in equidistribution in intervals of unit length.

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In this talk, we will be interested in equidistribution in intervals of unit length.

For every $m \in \mathbb{Z}$, define **"Weyl limits":**

$$c_m := \lim_{n\to\infty} \frac{1}{\#A_n} \sum_{t\in A_n} e(mt).$$

 

(I.J Schoenberg, 1903 - 1990) (N. Wiener, 1926 - 1964)
(Source: wikipedia.org)

Theorem 7 (Wiener-Schoenberg)

A_n is equidistributed with respect to some positive continuous measure if and only if the Weyl limits exist for every integer m and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{|m|\leq N} |c_m|^2 = 0.$$

(P. Erdos, 1913- 1996) (Turán, 1910- 1976)
(Source: wikipedia.org)

Theorem 8 (Erdös-Turán inequality, 1949)

For any positive integer M and subinterval I of $[0,1]$, there exist constant c_1 and c_2 such that

$$|\#\{n \leq N : x_n \in I\} - N\mu(I)| \leq \frac{c_1 N}{M+1} + c_2 \left| \sum_{m=1}^{M} \frac{1}{m} e(mx_n) \right|.$$

(H.L. Montgomery)
(Source: wikipedia.org)

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Theorem 9

For any positive integer M and $[a, b] \subset [0, 1]$, we have

$$|\#\{n \leq N : x_n \in [a, b]\} - N(b-a)|$$

$$\leq \frac{N}{M+1} + \sum_{1 \leq |m| \leq M} \left(\frac{1}{M+1} + min \left(b-a, \frac{1}{\pi|m|} \right) \right) |\sum_{m \leq M} e(mx_n)|.$$

Part-II: Distribution of gaps between elements of equidistributed sequences.

(Van der Corput, 1890 – 1975)
(Source: wikipedia.org)

Theorem 10 (Van der Corput,1931)

If for each positive integer s, the sequence $\{x_{n+s} - x_n\}$ is equidistributed (mod 1), then the sequence $\{x_n\}$ is equidistributed (mod 1).

Let us consider the following classical question:

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Is the converse of Van der Corput's result true?

Answer: NO!

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Answer: NO!

Example: $\{n\theta\} \pmod 1$, θ is irrational.

For any natural number N and $0 < b < 1$, define,

$$\begin{aligned} A_b(N) &:= \{n\theta \pmod 1 : n\theta \pmod 1 \leq b,\ 1 \leq n \leq N\} \\ &\subset \{\{n\theta\}\ (mod\ 1) : n \in \mathbb{N}\}, \end{aligned}$$

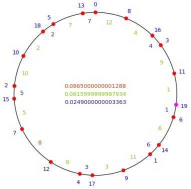
and write them as increasing order as follows:

$$A_b(N) = \{0 < x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_N \leq b < 1\}$$

(H. Steinhaus, 1887 – 1972)
(Source: wikipedia.org)

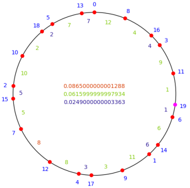
In 1957, Steinhaus conjectured the following:

$$\#\{x_{i+1} - x_i : 1 \leq i \leq N\} \leq 3.$$



$$\left(\theta = 1 - \frac{1+\sqrt{5}}{2}\right)$$

(Source: idiot's Blog)



$(\theta = 1 - \frac{1+\sqrt{5}}{2})$
(Source: idiot's Blog)

(Vera Sós)
(Source: wikipedia.org)

The first proof is due to Vera Sós.

Notations/Assumptions

For $1 \leq i \leq r$, Let $\{A_{i_n}\} = \{\pm x_i\} \subset [-\frac{1}{2}, \frac{1}{2}]$ be sequences of finite multisets with $\# A_{i_n} \to \infty$.

For every $m \in \mathbb{Z}$, let $c_{i_m}, 1 \leq i \leq r$ denote the m^{th} "Weyl limit" of $\{A_{i_n}\}$ respectively.

Let us assume that $\{A_{i_n}\}$ are equidistributed in $[-\frac{1}{2}, \frac{1}{2}]$ with respect to the measure $F_i(x)dx$ respectively, where

$$F_i(x) = \sum_{m=-\infty}^{\infty} c_{i_m} e(mx).$$

and for every $m \in \mathbb{Z}$,

$$c_{i_m} := \lim_{n \to \infty} \frac{1}{\# A_{i_n}} \sum_{t \in A_{i_n}} e(mt).$$

Here onwards, let us denote $\{x\}$ as fractional part of x.

Observation 11

Let C_m be the m^{th} Weyl limit of the family

$$\{\{\pm x_1 \pm x_2 \pm \cdots \pm x_r\},\ \pm x_i \in A_{i_n},\ 1 \leq i \leq r\}$$

that is for $m \in \mathbb{Z}$,

$$C_m := \lim_{n\to\infty} \frac{1}{\prod_{i=1}^{r} \# A_{i_n}} \sum_{\substack{x_i \in A_{i_n} \\ 1 \leq i \leq r}} e(m\{\pm x_1 \pm x_2 \pm \cdots \pm x_r\}).$$

Then the Weyl limit

$$C_m = \prod_{i=1}^{r} c_{i_m}. \tag{1}$$

Theorem 12

If

$$\sum_{m=-\infty}^{\infty} |c_{i_m}|^2 < \infty \quad \text{for all } 1 \leq i \leq r, \tag{2}$$

then the family

$$\{\{\pm x_1 \pm x_2 \pm \cdots \pm x_r\},\ \pm x_i \in A_{i_n}\}$$

is equidistributed in $[0, 1]$ *with respect to the measure*

$$\mu = F(x)dx,$$

where

$$F(x) = F_1 * F_2 * \cdots * F_r(x).$$

Notations/Assumptions

Let C_m be the Weyl limit of the family

$$\{\{\pm x_1 \pm x_2 \pm \cdots \pm x_r\},\ \pm x_i \in A_{i_n},\ 1 \le i \le r\}.$$

Let $I = [a, b] \subset [0, 1]$, and $V_n = \prod_{i=1}^{r} \# A_{i_n}$.

Let $\underline{x} = \{\pm x_1 \pm x_2 \pm \cdots \pm x_r\}$.

Define,

$$N_I(V_n) := \# \left\{ (x_1, x_2, ..., x_r) \in A_{1_n} \times A_{2_n} \times \cdots \times A_{r_n} : \underline{x} \in I \right\}.$$

and

$$D_{I,V_n}(\mu) := \frac{1}{V_n} |N_I(V_n) - V_n \mu(I)|.$$

A Variant of the Erdös-Turán inequality

Theorem 13

If $\sum_{m=-\infty}^{\infty} |c_{i_m}|^2 < \infty$ *for all* $1 \leq i \leq r$. *Then, for any positive integer* M *and any* $I = [a, b] \subset [0, 1]$, *we have*

$$|D_{I,V_n}(\mu)| \leq \frac{V_n}{M+1} + \sum_{|m| \leq M} \left(\frac{1}{M+1} + min\left(b - a, \frac{1}{\pi |m|} \right) \right)$$

$$\left(\left| \prod_{i=1}^{r} \sum_{x_i \in A_{i_n}} e(mx_i) - \prod_{i=1}^{r} \# A_{i_n} c_{i_m} \right| \right),$$

where $\mu = F_1 * F_2 * \cdots * F_r(x)dx$.

PART-III: Sato-Tate conjecture and beyond.

- $S(N, k)$: space of cusp forms of weight k and level N.
- $s(N, k)$: dimension of $S(N, k)$.

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- $s(N,k)$: dimension of $S(N,k)$.
- $T_n : S(N,k) \to S(N,k)$, n-th Hecke operator.
- $T'_p = \frac{T_p}{p^{(k-1)/2}}$.

From the **Ramanujan-Petersson conjecture:** (proved by Deligne, 1974.) we know that, for any prime p not dividing N, the eigenvalues of normalized Hecke operator $T_p^{'}$ say $\{a_{p,i,N}, 1 \leq i \leq s(N,k)\}$ lie in $[-2, 2]$.

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How are they distributed?

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Horizontal: Fix $S(N,k)$ and vary primes p.

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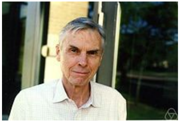
How are they distributed?

Horizontal: Fix $S(N,k)$ and vary primes p.

Vertical: Fix a prime p and vary $S(N,k)$.

Sato-Tate conjecture

(Mikio Sato)



(John Tate)

(Source: wikipedia.org)

Fix $S(N,k)$. Let f be a fixed normalized Hecke eigenform of $S(N,k)$ and let $a_p(f)$ denote the eigenvalue of $T_p^{'}$ relative to f. For every interval $[\alpha,\beta] \subset [-2,2]$,

$$\lim_{n\to\infty} \frac{\#\{p \leq n : \ a_p(f) \in [\alpha,\beta]\}}{\pi(n)} = \int_\alpha^\beta \mu_\infty,$$

where

$$\mu_\infty = \begin{cases} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\, dx \text{ if } x \in [-2,2], \\ 0 \text{ otherwise.} \end{cases}$$

In a series of papers by Richard Taylor, Michael Harris, Nick Shepherd-Barron, David Geraghty, Laurent Clozel and Tom Barnet-Lamb, this conjecture has now been proved for the cases when $k \geq 2$ and level $N \geq 1$.

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(Source: wikipedia.org)

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(J.P Serre)
(Source: wikipedia.org)

In the year 1997 Serre studied the "vertical" Sato-conjecture by fixing a prime p and varying N and k.

Theorem 14 (Serre, 1997)

Let N_λ, k_λ be positive integers such that k_λ is even, $N_\lambda + k_\lambda \to \infty$ and p is a prime not dividing N_λ for any λ. Then the family of eigenvalues of the normalized pth Hecke operator

$$T_p^{'}(N_\lambda, k_\lambda) = \frac{T_p(N_\lambda, k_\lambda)}{p^{\frac{k_\lambda - 1}{2}}}$$

is equidistributed in the interval $\Omega = [-2, 2]$ with respect to the measure

$$\mu_p := \frac{p+1}{\pi} \frac{\sqrt{1 - \frac{x^2}{4}}}{(p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 - x^2} dx.$$

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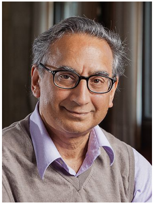
Theorem 15 (Serre, 1997)

Let $S^{'}(N, k)$ be the space of normalised Hecke eigen forms of weight k and level N. For any positive integer d and fixed k,

$$\#\{f \in S^{'}(N, k) : [K_f : \mathbb{Q}] \leq d\} = o(s(N, k)) \text{ as } N \to \infty,$$

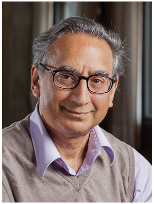
where $K_f(n) = \mathbb{Q}(\{a_n(f)\}_{n \geq 1})$.



(Ram Murty)
(Source: gregblack.ca)

(Kaneenika Sinha)
(Source: IISER Pune)

In the year 2009 Murty and Sinha give explicit estimate on the rate of convergence. They prove the following theorem



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Theorem 16 (Murty-Sinha, 2009)

Let p be a fixed prime. Let $\{(N,k)\}$ be a pairs of positive integers such that k is even, p is coprime to N. For an interval $[\alpha,\beta] \subset [-2,2]$

$$\frac{1}{s(N,k)} \sharp\left\{1 \le i \le s(N,k) : a_i(p) \in [\alpha,\beta]\right\} = \int_\alpha^\beta \mu_p + O\left(\frac{\log p}{\log kN}\right).$$

Theorem 17 (Murty-Sinha, 2009)

$$\#\{f \in S^{'}(N,k) : [K_f : \mathbb{Q}] \le d\}$$

$$\le d^2 \prod_{i=1}^{d} \left(2 \binom{d}{i} ((2p)^{\frac{k-1}{2}})^i + 1 \right) \left(\frac{3s(N,k) \log\, p}{\log\, kN} + 63(kN \frac{\log\, p}{\log kN}) \right).$$

Let $a_{p,i,N} = 2\cos\theta_{p,i,N}$ for some $\theta_{p,i,N} \in [0,\pi]$ and

$$\theta_{i_1,i_2,..,i_r} = \frac{\pm\theta_{p,i_1,N} \pm \theta_{p,i_2,N} \pm \cdots \pm \theta_{p,i_r,N}}{2\pi}.$$

Theorem 18

Let N be a positive integer and p a prime number coprime to N. For an interval $[\alpha, \beta] \subset [0,1]$, $1 \leq r \leq s(N,k)$,

$$\frac{1}{(s(N,k))^r} \# \{1 \leq i_1, i_2, ..., i_r \leq s(N,k) : \{\theta_{i_1, i_2, .., i_r}\} \in [\alpha, \beta]\}$$

$$= \int_{[\alpha,\beta]} \nu_p + O\left(\frac{\log\ p}{\log\ kN}\right),$$

$$\textit{where,} \quad \nu_p = F(x) * F(x) * \cdots * F(x) dx$$

$$\textit{and} \quad F(x) = 4(p+1) \frac{\sin^2 2\pi x}{(p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 - \cos^2 2\pi x}.$$

Here the implied constant is effectively computable.

The following theorem can be deduced from the above theorem

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Theorem 19

For any $\alpha \in [0, 1]$,

$$\sharp \left\{ 1 \leq i_1, i_2, ..., i_r \leq s(N, k) : \left\{ \frac{\pm\theta_{p,i_1,N} \pm \theta_{p,i_2,N} \pm \cdots \pm \theta_{p,i_r,N}}{2\pi} \right\} = \alpha \right\}$$

$$= O\left((s(N, k))^r \left(\frac{\log\ p}{\log\ kN} \right) \right),$$

where the implied constant is effectively computable.

In the above theorem for $r = 2$, we have an interesting consequence.

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Theorem 20

For any $\alpha \in [0,1]$

$$\sharp\left\{(i,j) : \left\{\frac{\pm\theta_{p,i,N} \pm \theta_{p,j,N}}{2\pi}\right\} = \alpha\right\} = O\left((s(N,k))^2\left(\frac{\log\, p}{\log\, kN}\right)\right).$$

$$\text{Since,} \quad \sharp\left\{(i,j) : \left(\frac{\pm\theta_{p,i,N} \pm \theta_{p,j,N}}{2\pi}\right) = 0\right\}$$

$$\leq \sharp\left\{(i,j) : \left\{\frac{\pm\theta_{p,i,N} \pm \theta_{p,j,N}}{2\pi}\right\} = 0\right\},$$

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Theorem 21

$$\sharp\{(i,j) : (\pm\theta_{p,i,N} \pm \theta_{p,j,N}) = 0\} = \mathrm{O}\left((s(N,k))^2 \left(\frac{\log\ p}{\log\ kN}\right)\right).$$

$$\text{Since,} \quad \sharp\left\{(i,j) : \left(\frac{\pm\theta_{p,i,N} \pm \theta_{p,j,N}}{2\pi}\right) = 0\right\}$$

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Remark 22

The above result recover main result of Murty-Srinivas.

Remark 23

The above result give some evidence towards Maeda and Tsaknias conjecture.

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The above result give some evidence towards Maeda and Tsaknias conjecture.

In 1997, Maeda predicts that for $N = 1$,

$$\prod_{i=1}^{s(N,k)} (x - a_{p,i,1}) \text{ is irreducible over } \mathbb{Q}.$$

For higher level Tsaknias predicts that the above polynomial is

product of bounded number of irreducible polynomials over $\mathbb{Q}$.

Sketch of proof

For the family $\left\{\pm\frac{\theta_{p,i,N}}{2\pi}\right\}$,

- $$\sum_{i=1}^{s(N,k)} e\left(\pm m\theta_{p,i,N}\right) = \sum_{i=1}^{s(N,k)} 2\cos m\theta_{p,i,N}.$$

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  For $m = 1$,

- $$\sum_{i=1}^{s(N,k)} 2\cos\theta_{p,i,N} = TrT_p^{'}.$$

Sketch of proof

For the family $\left\{\pm\frac{\theta_{p,i,N}}{2\pi}\right\}$,

- $$\sum_{i=1}^{s(N,k)} e\left(\pm m\theta_{p,i,N}\right) = \sum_{i=1}^{s(N,k)} 2\cos m\theta_{p,i,N}.$$

  For $m = 1$,

- $$\sum_{i=1}^{s(N,k)} 2\cos\theta_{p,i,N} = TrT_p^{'}.$$

  For $m \geq 2$,

- $$\sum_{i=1}^{s(N,k)} 2\cos m\theta_{p,i,N} = TrT_{p^m}^{'} - TrT_{p^{m-2}}^{'}.$$

- Hilbert modular forms (Arthur trace formula).
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- Primitive Maass forms (Kuznetsov trace formula).
  Y.-K. Lau and Y. Wang, *Quantitative version of the joint distribution of eigenvalues of the Hecke operators*, J. Number Theory 131 (2011), 2262–2281.
- Certain families of Elliptic curve.
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Joint Sato-Tate conjecture

Let f_1 and f_2 be two Hecke eigenforms such that $f_1(p)$ is not a character multiple of $f_2(p)$. For any rectangle $I \subset [-2,2]^2$,

$$\frac{1}{\pi(x)}\#\{p \le x : (a_p(f_1), a_p(f_2)) \in I\}$$

$$= \int_I d\mu \times d\mu,$$

where $d\mu$ is the Sato-Tate measure.

Theorem 24 (Murty, —, 2016)

Let f_1, f_2 be normalized Hecke eigenforms of weight k_1, k_2 respectively such that $f_i(z) = \sum_{n\geq 1} \frac{a_n(f_i)}{n^s}$. Suppose that atleast one of f_1, f_2 is not of CM type. Write,

$$a_p(f_i) = b_p(f_i) p^{\frac{k-1}{2}}.$$

If

$$\limsup_{x\to\infty} \frac{\#\{p \leq x : \ b_p(f_1) = b_p(f_2)\}}{x/\log x} > 0,$$

then $f_1 = f_2 \otimes \chi$ for some Dirichlet character χ.

Remarks

- Since Dirichlet characters are well understood, the above theorem classifies eigenforms under the above restriction.
- This theorem also proves $k_1 = k_2$.
- Rajan (1998) proved the above result when $k_1 = k_2$.
- As a corollary, he obtained the following result
  Let E_1 and E_2 be two elliptic curves over $\mathbb{Q}$. If

$$\limsup_{x \to \infty} \frac{\{p : \#E_1(\mathbb{F}_p) = \#E_2(\mathbb{F}_p)\}}{x / \log x} > 0,$$

  then E_1 and E_2 are isogenous after base change.
- Recently, Kulkarni, Patankar and Rajan extend the above result for number fields (using Galois theory, Chebotarev density theorem etc).
- Using modularity theorem, we get the above result over $\mathbb{Q}$.

Sketch of proof

To prove the above theorem, we need the following proposition.

Proposition 25

Let $0 < \delta < \pi$. *Let* $f_\delta(x)$ *be the "tent" function defined on* $[-\pi, \pi]$ *be given by*

$$f_\delta(x) = \begin{cases} 1 - |x|/\delta & \textit{if } |x| \leq \delta, \\ 0 & \textit{if } |x| > \delta. \end{cases}$$

Then, for any $M \geq 1$, *we have*

$$f_\delta(x) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{M} \frac{1 - \cos n\delta}{\pi n^2 \delta} \cos nx + O\left(\frac{1}{M\delta}\right),$$

where the implied constant is absolute.

Note that

$$\#\{p \le x : \theta_p^{(1)} = \theta_p^{(2)}\} \le \sum_{p \le x} f_\delta(\theta_p^{(1)} - \theta_p^{(2)}) + f_\delta(\theta_p^{(1)} + \theta_p^{(2)}).$$

$$\le \frac{\delta \pi(x)}{\pi} + 4 \sum_{n=1}^{M} \frac{1 - \cos n\delta}{\pi n^2 \delta} \sum_{p \le x} \cos n\theta_p^{(1)} \cos n\theta_p^{(2)} + O\left(\frac{\pi(x)}{M\delta}\right)$$

upon using the trigonometric identity

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$$

and

$$2 \cos n\theta = \frac{\sin(n+1)\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta}.$$

proof continued

we can rewrite our sum as

$$\sum_{n=2}^{M} \frac{1-\cos n\delta}{\pi n^2 \delta} \times \sum_{p \le x} \left(\left(\frac{\sin(n+1)\theta_p^{(1)}}{\sin \theta_p^{(1)}} - \frac{\sin(n-1)\theta_p^{(1)}}{\sin \theta_p^{(1)}} \right) \right.$$

$$\left. \times \left(\frac{\sin(n+1)\theta_p^{(2)}}{\sin \theta_p^{(2)}} - \frac{\sin(n-1)\theta_p^{(2)}}{\sin \theta_p^{(2)}} \right) \right).$$

proof continued

Now to complete the proof it is sufficient to prove the following Proposition.

Proposition 26

If f_1, f_2 are normalized Hecke eigenforms, with at least one not of CM type, such that $f_1 \neq f_2 \otimes \chi$ for some Dirichlet character χ, then for any positive integers m, n,

$$\sum_{p \leq x} \frac{\sin(m+1)\theta_p^{(1)}}{\sin \theta_p^{(1)}} \frac{\sin(n+1)\theta_p^{(2)}}{\sin \theta_p^{(2)}} = o(x/\log x),$$

as x tends to infinity. Here, the summation is over primes.

Joint Sato-Tate distribution for two Hecke eigenforms

Theorem 27

Let f_1 and f_2 be two Hecke eigenforms such that $f_1(p)$ is not a character multiple of $f_2(p)$. For any rectangle $I \subset [-2, 2]^2$,

$$\frac{1}{\pi(x)} \#\{p \leq x : (a_p(f_1), a_p(f_2)) \in I\}$$

$$= \int_I d\mu \times d\mu,$$

where $d\mu$ is the Sato-Tate measure.

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THANK YOU