p-Modular Representations of SL_3 over a Finite Field

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Modular Representation Theory

Let G be a finite group.

- The representation theory of G over an algebraically closed field $\mathbb{F}$ with characteristic p, a prime, is the **modular representation theory** of the group algebra $\mathbb{F}G$.
- An $\mathbb{F}G$-module M is **simple** if M is non-zero and the only submodules of M are $\{0\}$ and M.
- The group algebra $\mathbb{F}G$ is **semisimple** or *completely reducible* if all $\mathbb{F}G$-modules M can be expressed as

$$M = \bigoplus_i S_i$$

  where each S_i is simple.
- (Maschke) A group algebra $\mathbb{F}G$ is **semisimple**, if and only if $p \nmid |G|$.

Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_q$, where $\mathbb{F}_q$ is the Galois field of q elements and q is a power of p.

- $\mathbf{GL_n}(\mathbb{F}_\mathbf{q})$ is the group of all invertible matrices of order n.
- **Order**: $|GL_n(\mathbb{F}_q)| = \prod_{i=0}^{n-1}(q^n - q^i)$
- The $\mathbf{SL_n}(\mathbb{F}_\mathbf{q})$ is its normal subgroup consisting of the matrices of determinant 1.
- **Order**: $|SL_n(\mathbb{F}_q)| = \frac{1}{q-1}|GL_n(\mathbb{F}_q)|$.
- $p \mid |SL_n(\mathbb{F}_q)| \Rightarrow$ The group algebra $\mathbb{F}G$ loses its semisimplicity!

Goal

Let G be the group $SL_n(\mathbb{F}_q)$.

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  Reduction mod p of the ordinary representations gives the **representations over the field $\mathbb{F}$**.

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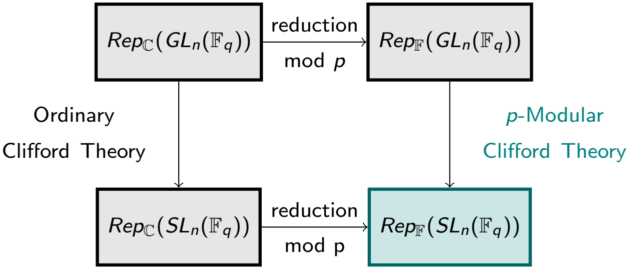
- **Independent of ordinary representations?**
  Bonnafe's approach for $SL_2(\mathbb{F}_q)$ to compute the "vector spaces" explicitly.
- **Goal**: An explicit description of the modular representations of $SL_n(\mathbb{F}_q)$.

Our approach

- Almost everything about $GL_n(\mathbb{F}_q)$ is known.



Subgoal/Current Project

An explicit description of the modular representations of $SL_3(\mathbb{F}_q)$.

Clifford Theorem

Let N be the normal subgroup of a group G.

Theorem (Clifford)

- *A simple $\mathbb{F}G$-module V is semisimple as an $\mathbb{F}N$-module.*
- $V \cong Ind_{I_G(W)}^{G}(\tilde{W})$ *as $\mathbb{F}G$-modules.*

Here $I_G(\sigma) = \{g \in G :^g \sigma \sim \sigma\} < G$ is the **inertia group**.

p-Modular Clifford Theorem (B.)

- Let p a prime such that $p \mid |N|$. Let $\sigma \in \hat{N}$, $\theta \in \hat{G}(\sigma)$ be an irreducible p-modular representations.

Theorem

Let R be the set of representatives for the left $I_G(\sigma)$-cosets in G and ℓ denotes the multiplicity of σ in $Res_N^G \theta$. Then

$$Res_N^G \theta = \ell \bigoplus_{r \in R} {}^r\sigma$$

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- Define a map from $\hat{I}(\sigma)$ to $\hat{G}(\sigma)$ by $\phi \mapsto Ind_I^G(\phi)$.

Clifford Correspondence

The map is a bijection.

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- Let G/N be abelian.

Theorem

$Ind_N^G\sigma = \bigoplus_{\phi \in \widehat{G/N}} (\theta \otimes \bar{\phi})$

Example: $SL_2(\mathbb{F}_p)$

- $0 \leq k \leq p-1$ and $0 \leq r \leq p-2$
- **$\mathbf{Pol_k(r)}$** is the space of polynomial such that for all $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $P(x,y) \in Pol_k(r)$,

$$\rho_{Pol_k(r)} \cdot P(x,y) = P(ax+cy, bx+dy) \otimes det(g)^r.$$

- **$\mathbf{Pol_k(r)}$** are the irreducible representations of $GL_2(\mathbb{F}_p)$.
- There are p distinct irreducible representations of $SL_2(\mathbb{F}_p)$, up to isomorphism, given by **$\mathbf{Pol_k}$**.

Then,

Observe

$Ind_N^G Pol_k = \bigoplus_{r=1}^{p-2} Pol_k(r)$, $0 \leq k \leq p-1$.

$SL_3(\mathbb{F}_q)$: Notations

- Let $\mathbf{G} = SL_3(\mathbb{F})$, and G denotes its fixed point subgroup $SL_3(\mathbb{F}_q)$ under the Frobenius endomorphism.
- Let V_3 denotes a natural rational $\mathbf{G}$-module of dimension 3. Then $V_3 = \mathbb{F}^3$ and is equipped with standard action of $\mathbf{G}$.
- For $n \in \mathbb{N}$, let $\Delta(n)$ be a rational $\mathbf{G}$-module defined as

$$\Delta(n) = Sym^n(V_3)$$

  where $Sym^n(V_3)$ denotes the n-th symmetric power of the vector space V_3.

$SL_3(\mathbb{F}_q)$: Construction

- We denote by (x, y, z) the canonical basis of V_3 and $g = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ such that
  $g \cdot x = a_1 x + b_1 y + c_1 z,$
  $g \cdot y = a_2 x + b_2 y + c_2 z,$
  $g \cdot z = a_3 x + b_3 y + c_3 z.$
- We have
$$\Delta(n) = \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{i} \mathbb{F} x^{n-i} y^{i-j} z^{j}.$$
- Let **L(n)** denotes the **G**-submodule of $\Delta(n)$ generated by x^n.

Main Results

- **Lemma 1**
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- **Lemma 3**
  If $m, n \geq 0$ then $L(m) = L(n)$ if and only if $m = n$.
- (Brauer-Nesbitt) The number of irreducible p-modular representations is equal to the number of the p-regular conjugacy classes of the group.

Hence some of these representations i.e. some of the simple $\mathbb{F}G$-modules, can be *explicitly* described as follows:

Theorem (B.)

For $0 \leq n \leq q-1$, the family $Res_G^{\mathbf{G}} L(n)$ is a set of representatives of the isomorphism classes of simple $\mathbb{F}G$-modules.

Ingredients of the proof:

- Let $\mathbb{F}[\mathbf{G}]$ be the group algebra of regular functions such that for $g, x \in \mathbf{G}$

$$(g \cdot f)(x) = f(g^{-1}x),$$

- Set

$$\Delta'(n) = \{f \in \mathbb{F}[\mathbf{G}] | \forall (g, b) \in \mathbf{G} \times \mathbf{B}, f(gb) = \tilde{\varepsilon}^{-n}(b) f(g)\}$$

Last slide!

- Let V be a simple rational **G**-module. Then there exists a non-zero morphism
$$V \to \Delta'(n)$$
- Moreover, for $\Delta'(n) \neq 0$, $\Delta'(n) \cong \Delta(n)$.
- Therefore, there exists a correspondence between V and $L(n)$, and restricting $0 \leq n \leq q-1$ gives the above result. $\square$

Further Questions:

- Is the above set exhaustive?
- Generalization of the above result to $SL_n(F_q)$?

Thank you for listening!