DETERMINING HILBERT MODULAR FORMS USING SQUARE-FREE COEFFICIENTS

(Joint work with Rishabh Agnihotri)

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$$L\left(\frac{1}{2}, f \otimes \chi_d\right) = L\left(\frac{1}{2}, g \otimes \chi_d\right)$$

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Kohnen (1992 Math. Ann.) : Let f, g be two non-zero Hecke eigenforms in $S^+_{k+\frac{1}{2}}(4)$ with Fourier coefficients $a(n)$ and $b(n)$ respectively and suppose that $a(|D|) = b(|D|)$ for all fundamental discriminants D such that $(-1)^k D > 0$. Furthermore suppose that $\lambda_2 = \mu_2$, where λ_2 (respectively μ_2) denote the eigenvalues of f (respectively g) under the Hecke operator $T^+_{k+\frac{1}{2}}(4)$. Then $f = g$.

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Let F be a totally real number field with narrow class number 1 of degree r over $\mathbb{Q}$. We fix (once and for all) embeddings $\sigma_1, \sigma_2, \ldots, \sigma_r$ of F into the reals. For any σ_i and $\xi \in F$, it will also be convenient to denote $\sigma_i(\xi)$ as ξ_i.

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We embed the unit group U of $\mathcal{O}_F$ inside $SL_2(\mathcal{O}_F)$ as

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We adopt the following multi-index notation.

1. We shall embed $F \hookrightarrow \mathbb{R}^r \hookrightarrow \mathbb{C}^r$ via its various embeddings $\{\sigma_i\}$. We shall also identify $\mathbb{C} \hookrightarrow \mathbb{C}^r$ via the diagonal embedding.
2. Addition and multiplication in $\mathbb{C}^r$ is point-wise. For $\boldsymbol{w} = (w_1, \dots, w_r)$, $\boldsymbol{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$ we let $\boldsymbol{w} \cdot \boldsymbol{z} \in \mathbb{C}$ be the sum $\sum_i w_i z_i$.

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Associated to each holomorphic Hilbert modular form is a weight $\boldsymbol{k} = (k_1, \ldots, k_r)$ of the form $\boldsymbol{m_k} + \delta_{\boldsymbol{k}}(\frac{1}{2}, \ldots, \frac{1}{2})$, where $\boldsymbol{m_k} \in \mathbb{Z}^r$ and $\delta_{\boldsymbol{k}} \in \{0,1\}$.

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Hilbert modular forms have a "Fourier expansion" given by

$$f(\boldsymbol{z}) = \sum_{\xi \in \mathcal{O}_F^+} \lambda_f(\xi)\, \xi^{\frac{\boldsymbol{k}-1}{2}}\, e^{(2-\delta_k)\pi\, i\, \xi \cdot \boldsymbol{z}}.$$

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We may associate the following L function to f,

$$L(s, f) \coloneqq \sum_{\xi \in \mathcal{O}_{\mathcal{F}}^+ / U^+} \frac{\lambda_f(\xi)}{\xi^s}$$

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Given a half integral weight Hecke eigenform f, and a square-free integer $\tau \in \mathcal{O}_F$, there exists an integral weight Hecke eigenform $\mathfrak{f}_\tau$, with Fourier coefficients given by $c(\xi, f_\tau)$ such that the following formal identity holds,

$$\sum_{\xi \in \mathcal{O}_F/U} c(\xi, \mathfrak{f}_\tau) M(\xi) = \left(\sum_{\xi \in \mathcal{O}_F/U} a_f(\tau\xi^2)\xi^{-\boldsymbol{m}} M(\xi) \right) \left(\sum_{\xi \in \mathcal{O}_F/U} \frac{\chi_\tau(\xi) M(\xi)}{N(\xi)} \right) = a_f(\tau) \prod_{\mathfrak{p}} \left(1 - \omega_{\mathfrak{p}} M(\mathfrak{p}) + \frac{M(\mathfrak{p}^2)}{N(\mathfrak{p})} \right)^{-1}$$

where $M(\xi) = M(\xi\mathcal{O}_{\mathcal{F}})$ is a formal symbol satisfying $M(\xi_1\xi_2) = M(\xi_1)M(\xi_2)$.

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where $M(\xi) = M(\xi\mathcal{O}_{\mathcal{F}})$ is a formal symbol satisfying $M(\xi_1\xi_2) = M(\xi_1)M(\xi_2)$.

We observe that the form $\mathfrak{f}_\tau / a_f(\tau)$ is defined independent of τ as long as $a_f(\tau) \neq 0$. We shall refer to this form as the Shimura lift of f.

Main Theorem

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Suppose that f and g are half-integral weight cuspidal Hecke eigenforms of varying weights and levels. Denote the normalized Fourier coefficients of f, g as λ_f and λ_g respectively. Suppose there exists $\kappa \neq 0$ such that $\lambda_f(\tau) = \kappa\lambda_g(\tau)$ for every square-free $\tau \in \mathcal{O}_F$. Then $f = \kappa g$.

An Intermediate Result

Theorem

Suppose f is a half integral weight cusp form of weight $\boldsymbol{k}$ and level $n\mathcal{O}_F$. Then, $L(s,f)$ converges absolutely for $Re(s) > 1$, continues analytically to the whole complex plane and satisfies a functional equation of the form $\Lambda(s,f) = \Lambda\big(1-s, f|_{W'(n)}\big)$ where

$$\Lambda(s,f) \coloneqq D_F^s N(n)^{\frac{s}{2}} (2\pi)^{-rs} \prod_{j=1}^{r} \Gamma\left(s + \frac{k_j - 1}{2}\right) L(s,f).$$

Sketch of Proof

Given f, g as in the statement of the theorem, define $C(s) \coloneqq \frac{\Lambda(s,f)}{\Lambda(s,g)}$.

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$$L(s,f) = L(2s,\mathfrak{f})\left(\sum_{\tau\in\mathcal{O}_{\mathcal{F}}^{+}/U^{+}} \frac{\lambda_f(\tau)}{N(\tau)^s} L\left(2s+\frac{1}{2},\chi_\tau\right)^{-1}\right).$$

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$$C(s) = \left(\frac{N(n_f)}{N(n_g)}\right)^{\frac{s}{2}} \left(\frac{N(n_g)}{N(n_f)}\right)^{s} 2^{\boldsymbol{n}-\boldsymbol{m}} \left(\prod_{i=1}^{r} \frac{\Gamma\left(\frac{2s+n_i+\frac{1}{2}}{2}\right)}{\Gamma\left(\frac{2s+m_i+\frac{1}{2}}{2}\right)}\right) \frac{\Lambda(2s,\mathfrak{f})}{\Lambda(2s,\mathfrak{g})}.$$

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Use the fact that $C(s)=\frac{\Lambda\left(1-s,f|_{W'(n_f)}\right)}{\Lambda\left(1-s,g|_{W'(n_g)}\right)}$. Without loss of generality, we may suppose that for every $1\le i,j\le r$, $m_i\neq n_j$. Furthermore, we may suppose that m_1 is the minimum of $\{m_i,n_j\}_{i,j}$. Substituting $2s = m_1+\frac{5}{2}$, we get

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Using the functional equation of the integral weight forms, and choosing s appropriately, we can show that $N(n_f) = N(n_g)$, $C(s) = C(s+1)$. This in particular proves that $C(s)$ is an entire function.

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Letting $s\to\infty$ along horizontal lines, we may show that $\mathrm{C}(s)$ is uniformly bounded above by 4, completing the proof.

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Further research

1. Try to remove the restriction of narrow class number 1.
2. Refinements in terms of smaller subsets of square-free integers.
3. An effective version?

References

Primary Reference

Rishabh Agnihotri and K. K., "*Determination of Hilbert modular forms using square-free coefficients*", arXiv : 2403.16012

Other References

Ritabrata Munshi, Math. Ann., 2010

Ritabrata Munshi and Jyoti Sengupta, Math. Ann. 2015

Ritabrata Munshi and Jyoti Sengupta, Forum Mathematicum, 2015

Alia Hamieh and Naomi Tanabe, Transactions of AMS, 2017

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Background on Hilbert Modular forms

Paul Garrett's book titled "*Holomorphic Hilbert modular forms*"

Shimura's Duke papers, 1978, 1983, 1993

Thank you