Unitarizability for the Metaplectic groups of small order

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36th Automorphic form Workshop, Oklahoma State University

May, 2024

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Formulations

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❖ We denote $\widetilde{P_S}$ as the inverse image of P_S in $\widetilde{Sp(n)}$. The Levi subgroup of $\widetilde{P_S}$ is the quotient of $\widetilde{GL(n_1)} \times \cdots \times \widetilde{GL(n_j)} \times \widetilde{Sp(n - |S|)}$.

Genuine Representation

The representation of $\widetilde{GL(k,F)}$ is obtained from tensoring a representation of $GL(k,F)$ with a genuine character of $\widetilde{GL(k,F)}$ defined by:

$$\chi_\psi : (g,\epsilon) \longmapsto \epsilon\gamma_\psi(\det g)^{-1}.$$

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Therefore for any irreducible representation π of $GL(k,F)$ the representation $\widetilde{\pi}$ of $\widetilde{GL(k,F)}$ is defined as $\widetilde{\pi}(g,\epsilon) = \pi(g) \times \chi_\psi(g,\epsilon)$.

Segment Representation of $\widetilde{Sp(n)}$

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❖ For non-zero $r_{\widetilde{P_S}}(\pi)$, let τ be an irreducible subquotient in $r_{\widetilde{P_S}}(\pi)$, so $\tau = \rho_1 \otimes \cdots \otimes \rho_i \otimes \rho$, where ρ_i's and ρ are the irreducible genuine cuspidal representaions of $\widetilde{GL(n_i)}$'s and $Sp\widetilde{(n - |S|)}$.

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❖ For any ρ_i as above we can define $e(\rho_i)$ to be the unitary exponent such that $\rho_i = \nu^{e(\rho_i)} \rho^u$. Where ν denotes the determinant character and ρ^u is the unitary cuspidal genuine representations.

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❖ Then π is called strongly negative (or negative representations) if and only if for every embedding $\pi \hookrightarrow \rho_1 \times \cdots \times \rho_i \times \rho$, we have the following : $\sum_{k=1}^{m} n_k e(\rho_k) < 0(\text{or} \leq 0)$, for all $m = 1, 2 \cdots i$.

Segment Representation of $\widetilde{Sp(n)}$

❖ For π, an irreducible admissible genuine representation of $\widetilde{Sp(n)}$, we denote by $r_{\widetilde{P_S}}(\pi)$, the normalized Jacquet module with respect to $\widetilde{P_S}$.

❖ For non-zero $r_{\widetilde{P_S}}(\pi)$, let τ be an irreducible subquotient in $r_{\widetilde{P_S}}(\pi)$, so $\tau = \rho_1 \otimes \cdots \otimes \rho_i \otimes \rho$, where ρ_i's and ρ are the irreducible genuine cuspidal representaions of $\widetilde{GL(n_i)}$'s and $\widetilde{Sp(n-|S|)}$.

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❖ For a cuspidal representation ρ of some $GL(m_\rho)$, we denote a segment Δ by a set of cuspidal representations $= \{\rho, \nu\rho, \cdots, \nu^{k-1}\rho\}$. Let $\langle\Delta\rangle$ be the unique irreducible representation of $\rho \times \nu\rho \times \cdots \times \nu^{k-1}\rho$.

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❖If σ is an irreducible admissible genuine representation of $\widetilde{Sp(n)}$, then there exist a sequence of segments (of genuine representations) $1, ..., k$ such that $e(\Delta_1) \geq \ \geq e(\Delta_k) > 0$ and a negative (genuine) representation σ_{neg} such that $\sigma \sim \langle\Delta_1, ..., \Delta_k; \sigma_{\text{neg}}\rangle$.

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Tadic structure formula

❖ Let

$$R^{\text{gen}} = \bigoplus R(\widetilde{GL(n)})_{\text{gen}},$$

where $R(.)_{\text{gen}}$ denotes the Grothendieck group of finite length, smooth, genuine representation of $\widetilde{GL(n)}$.

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❖ Analogously, using the Jacquet module for the maximal parabolics of $\widetilde{GL(n)}$, we can define the following comultiplication formula:

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❖ For $\pi \in R^{\text{gen}}$ and $\sigma \in R$, we have the following formula $\mu^*(\pi \rtimes \alpha) = M^*(\pi) \rtimes \mu^*(\alpha)$.

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methods

❖ The standard way of obtaining new unitarizable representations from old ones is by parabolic induction, which preserves unitarity (unitary parabolic induction).

❖ A second way of proving unitarizability is by considering families. If a continuous family of irreducible hermitian representations of a reductive groups G contains at least one unitarizable representation, then all representations in the family are unitarizable. Furthermore, if a continuous family of irreducible hermitian representations is parameterized by unbounded set of unramified parameters, then all the representations in the family are non-unitarizable.

Irreducibility for Principal series

❖ Let $\chi_{\psi,V}$ be the character of $\widetilde{GL(n,F)}$ dependent on V and Weil index γ_ψ, where ψ is a non-trivial additive character of F. Let $\zeta_1, \zeta_2, \cdots, \zeta_n$ be the unitary characters on $F^\times$. Then one can show that the following induced representation of $\widetilde{Sp(2n,F)}$:

$$\Pi = \chi_{\psi,V}\zeta_1 \times \chi_{\psi,V}\zeta_2 \times \cdots \times \chi_{\psi,V}\zeta_n \rtimes \omega_0,$$

is irreducible, where ω_0 is the representation of $\widetilde{Sp(0,F)}$.

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❖ Let $\zeta_1, \zeta_2, \cdots, \zeta_n$ be the characters (not necessarily unitary) of $F^\times$. Suppose we have the following conditions: for any $1 \leq i \leq n$: $\zeta_i \neq \nu^{\pm 1/2}\zeta$, where ζ is a quadratic character.

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Hermitian representation

Let $\langle \Delta_1, \Delta_2, \cdots, \Delta_k; \sigma_{\mathrm{neg}} \rangle$ be the unique Langlands subrepresentation of the induced representation with the segmentation. So $\langle \Delta_1, \Delta_2, \cdots, \Delta_k; \sigma_{\mathrm{neg}} \rangle$ is the hermitian if and only if:

$$\langle \Delta_1, \Delta_2, \cdots, \Delta_k; \sigma_{\mathrm{neg}} \rangle = \langle \alpha\overline{\Delta_1}, \alpha\overline{\Delta_2}, \cdots, \alpha\overline{\Delta_k}; \overline{\widetilde{\sigma_{\mathrm{neg}}}} \rangle$$

So σ_{neg} needs to be Hermitian as well. Here $\alpha = \chi_{\psi,V}^2$. Therefore $\alpha\overline{\chi_{\psi,V}} = \chi_{\psi,V}$

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❖ Let $\chi_i \in \widehat{F}^\times$, then for $s_1 \geq s_2 \geq s_3 \geq 0$
$\chi_\psi\chi_1\nu^{s_1} \times \chi_\psi\chi_2\nu^{s_2} \times \chi_\psi\chi_3\nu^{s_3} \rtimes \omega_0$ is irreducible except for the following cases:

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Contd.

Theorem

Let $\chi_1, \chi_2 \in \widehat{F^\times}$*, two nonnegative* $s_1 \geq s_2 \in \mathbb{R}$*, and* $\zeta \in \widehat{F^\times}$ *with* $\zeta^2 = 1_{F^\times}$*. The representations* $\chi_{\psi,V}\nu^{s_1}\chi_1 \times \chi_{\psi,V}\nu^{s_2}\chi_2 \rtimes sp_{\zeta,1}$ *and* $\chi_{\psi,V}\nu^{s_1}\chi_1 \times \chi_{\psi,V}\nu^{s_2}\chi_2 \rtimes \omega^{+}_{\psi_a,1}$ *are irreducible unless* $(s_1, \chi_1) = (3/2, \zeta)$*,* $(1/2, \zeta_1)$*, where* $\zeta_1^2 = 1_{F^\times}$*,* $(s_2, \chi_2) = (3/2, \zeta)$*,* $(1/2, \zeta_2)$*, where* $\zeta_2^2 = 1_{F^\times}$*, or* $(s_1, \chi_1) = (s_2 \pm 1, \chi_2)$*.*
Furthermore, if $(s_i, \chi_i) \neq (3/2, \zeta)$*,* $(s_i, \chi_i) \neq (1/2, \zeta_i)$ *for* $i = 1, 2$ *and* $(s_1, \chi_1) \neq (s_2 \pm 1, \chi_2)$*, then* $\chi_{\psi,V}\nu^{s_1}\chi_1 \times \chi_{\psi,V}\nu^{s_2}\chi_2 \rtimes sp_{\zeta,1}$

$$= \begin{cases} \langle \chi_{\psi,V}\nu^{1/2}\zeta; \chi_{\psi,V}\chi_1 \times \chi_{\psi,V}\chi_2 \rtimes \omega_0 \rangle & \text{if } s_1 = s_2 = 0, \\ \langle \chi_{\psi,V}\nu^{1/2}\zeta, \chi_{\psi,V}\nu^{s_1}\chi_1; \chi_{\psi,V}\chi_2 \rtimes \omega_0 \rangle & \text{if } s_2 = 0 \text{ and } s_1 \leq 1/2, \\ \langle \chi_{\psi,V}\nu^{1/2}\zeta, \chi_{\psi,V}\nu^{s_1}\chi_1, \chi_{\psi,V}\nu^{s_2}\chi_2; \omega_0 \rangle & \text{if } 0 < s_2 \leq s_1 \leq 1/2, \\ \langle \chi_{\psi,V}\nu^{s_1}\chi_1, \chi_{\psi,V}\nu^{1/2}\zeta; \chi_{\psi,V}\chi_2 \rtimes \omega_0 \rangle & \text{if } s_2 = 0 \text{ and } 1/2 \leq s_1, \\ \langle \chi_{\psi,V}\nu^{s_1}\chi_1, \chi_{\psi,V}\nu^{1/2}\zeta, \chi_{\psi,V}\nu^{s_2}\chi_2; \omega_0 \rangle & \text{if } 0 < s_2 \leq 1/2 \leq s_1, \\ \langle \chi_{\psi,V}\nu^{s_1}\chi_1, \chi_{\psi,V}\nu^{s_2}\chi_2, \chi_{\psi,V}\nu^{1/2}\zeta; \omega_0 \rangle & \text{if } s_2 > 1/2, \end{cases}$$

Theorem

And $\chi_{\psi,V}\nu^{s_1}\chi_1 \times \chi_{\psi,V}\nu^{s_2}\chi_2 \rtimes \omega^{+}_{\psi_a,1}$

$$= \begin{cases} \chi_{\psi,V}\chi_1 \times \chi_{\psi,V}\chi_2 \rtimes \omega^{+}_{\psi_a,1} & \text{if } s_1 = s_2 = 0, \\ \left\langle \chi_{\psi,V}\nu^{s_1}\chi_1; \chi_{\psi,V}\chi_2 \rtimes \omega^{+}_{\psi_a,1} \right\rangle & \text{if } s_2 = 0 \text{ and } s_1 > 0, \\ \left\langle \chi_{\psi,V}\nu^{s_1}\chi_1, \chi_{\psi,V}\nu^{s_2}\chi_2; \omega^{+}_{\psi_a,1} \right\rangle & \text{if } s_2 > 0. \end{cases}$$

We also have the following: Let $\chi, \chi_1 \in \widehat{F^\times}$. Suppose s is non-negative, $s_1 \geq s_2$ and $s_3 \in \mathbb{R}$. Suppose $\zeta, \zeta_1 \in \widehat{F^\times}$ such that $\zeta^2 = \zeta_1^2 = 1$. Then the representations $\chi_{\psi,V}\nu^s\chi\mathrm{St}_{GL(2)} \times \chi_{\psi,V}\nu^{s_3}\chi_1 \rtimes \omega_0$ and $\chi_{\psi,V}\nu^s\chi\mathbb{1}_{GL(2)} \times \chi_{\psi,V}\nu^{s_3}\chi_1 \rtimes \omega_0$ are irreducible except in the case where $(s_3, \chi_1) = (1/2, \zeta_1)$, $(s, \chi) = (1/2, \zeta)$ or $(1, \zeta)$ or $(0, \zeta)$, $(s_3, \chi_1) = (s \pm 1/2, \chi), (s \pm 3/2, \chi)$. Also if $\chi_{\psi,V}\nu^{s_1}\chi \times \chi_{\psi,V}\nu^{s_2}\chi \times \chi_{\psi,V}\nu^{s_3}\chi_1 \rtimes \omega_0$ is a representation of length 2, then we have $\chi_{\psi,V}\nu^s\chi\mathbb{1}_{GL(2)} \times \chi_{\psi,V}\nu^{s_3}\chi_1 \rtimes \omega_0$

$$= \begin{cases} \left\langle \chi_{\psi,V}\nu^s\chi\mathbb{1}_{GL(2)}, \chi_{\psi,V}\nu^{s_3}\chi_1; \omega_0 \right\rangle & \text{if } s_3 > 0 \\ \left\langle \chi_{\psi,V}\nu^s\chi\mathbb{1}_{GL(2)}; \chi_{\psi,V}\chi_1 \rtimes \omega_0 \right\rangle & \text{if } s_3 = 0 \end{cases}$$

Cont

Theorem

For the other case we have $\chi_{\psi,V}\nu^s\chi St_{GL(2)} \times \chi_{\psi,V}\nu^{s_3}\chi_1 \rtimes \omega_0 =$

$$\begin{cases} \left\langle \chi_{\psi,V}\nu^{s+1/2}\chi, \chi_{\psi,V}\nu^{s-1/2}\chi, \chi_{\psi,V}\nu^{s_3}\chi_1; \omega_0 \right\rangle & s-1/2 > s_3 \\ \left\langle \chi_{\psi,V}\nu^{s+1/2}\chi, \chi_{\psi,V}\nu^{1/2-s}\chi, \chi_{\psi,V}\nu^{s_3}\chi_1; \omega_0 \right\rangle & 1/2-s > s_3, s < 1/2 \\ \left\langle \chi_{\psi,V}\nu^{s+1/2}\chi, \chi_{\psi,V}\nu^{1/2-s}\chi; \chi_{\psi,V}\chi_1 \rtimes \omega_0 \right\rangle & s_3 = 0, \; and \; s < 1/2 \\ \langle \chi_{\psi,V}\nu\chi; \chi_{\psi,V}\chi \times \chi_{\psi,V}\chi_1 \rtimes \omega_0 \rangle & s_3 = 0 \; and \; s = 1/2 \\ \langle \chi_{\psi,V}\nu\chi, \chi_{\psi,V}\nu^{s_3}\chi_1; \chi_{\psi,V}\chi \rtimes \omega_0 \rangle & 1 > s_3 > 0 \; and \; s = 1/2 \\ \left\langle \chi_{\psi,V}\nu^{s+1/2}\chi, \chi_{\psi,V}\nu^{s_3}\chi_1, \chi_{\psi,V}\nu^{s-1/2}\chi; \omega_0 \right\rangle & s+1/2 > s_3 > s-1/2 \\ \left\langle \chi_{\psi,V}\nu^{s_3}\chi, \chi_{\psi,V}\nu^{s+1/2}\chi_1, \chi_{\psi,V}\nu^{s-1/2}\chi; \omega_0 \right\rangle & s_3 > s+1/2, s > 1/2 \\ \left\langle \chi_{\psi,V}\nu^{s_3}\chi_1, \chi_{\psi,V}\nu^{s+1/2}\chi; \chi_{\psi,V}\chi \rtimes \omega_0 \right\rangle & s_3 > s+1/2, s = 1/2 \\ \left\langle \chi_{\psi,V}\nu^{s_3}\chi, \chi_{\psi,V}\nu^{s+1/2}\chi_1, \chi_{\psi,V}\nu^{1/2-s}\chi; \omega_0 \right\rangle & s_3 > s+1/2, s < 1/2 \end{cases}$$

Exceptional cases

They are listed as below:

$$\chi_{\psi,V}\zeta\nu^{3/2} \times \chi_{\psi,V}\zeta\nu^{1/2} \times \chi_{\psi,V}\zeta\nu^{1/2} \rtimes \omega_0.$$
$$\chi_{\psi,V}\zeta\nu^{3/2} \times \chi_{\psi,V}\zeta\nu^{3/2} \times \chi_{\psi,V}\zeta\nu^{1/2} \rtimes \omega_0.$$
$$\chi_{\psi,V}\zeta\nu^{5/2} \times \chi_{\psi,V}\zeta\nu^{3/2} \times \chi_{\psi,V}\zeta\nu^{1/2} \rtimes \omega_0.$$
$$\chi_{\psi,V}\zeta\nu \times \chi_{\psi,V}\zeta\nu \times \chi_{\psi,V}\zeta\nu^{-1/2} \rtimes \omega_0$$
$$\chi_{\psi,V}\zeta\nu \times \chi_{\psi,V}\zeta \times \chi_{\psi,V}\zeta \rtimes \omega_0$$
$$\chi_{\psi,V}\zeta\nu^{2} \times \chi_{\psi,V}\zeta\nu \times \chi_{\psi,V}\zeta \rtimes \omega_0$$
$$\chi_{\psi,V}\zeta\nu \times \chi_{\psi,V}\zeta \times \chi_{\psi,V}\zeta\nu^{-1} \rtimes \omega_0$$
$$\chi_{\psi,V}\zeta\nu^{1/2} \times \chi_{\psi,V}\zeta\nu^{-1/2} \times \chi_{\psi,V}\zeta\nu^{-1/2} \rtimes \omega_0.$$
$$\chi_{\psi,V}\zeta\nu^{1/2} \times \chi_{\psi,V}\zeta\nu^{1/2} \times \chi_{\psi,V}\zeta\nu^{-1/2} \rtimes \omega_0.$$
$$\chi_{\psi,V}\zeta\nu^{3/2} \times \chi_{\psi,V}\zeta\nu^{1/2} \times \chi_{\psi,V}\zeta\nu^{-1/2} \rtimes \omega_0.$$
$$\chi_{\psi,V}\zeta\nu^{1/2} \times \chi_{\psi,V}\zeta\nu^{-1/2} \times \chi_{\psi,V}\zeta\nu^{-3/2} \rtimes \omega_0.$$

Rank one reducibility

These results are obtained from thw work of Muic and Hanzer (2011)

- The representation

$$\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(3)}} \chi_{V,\psi} \chi \nu^s \otimes \sigma$$

reduces for $s = \{1/2, 3/2, 5/2\}$, depending on the Theta lift of σ.

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reduces for $s = 1/2$.

- The representation

$$\mathrm{Ind}_{\widetilde{P_2}}^{\widetilde{Sp(3)}} \chi_{V,\psi} \chi \nu^s \otimes \sigma$$

reduces for $s = \{0, 1/2, 1, 3/2\}$.

THANK YOU!