# Stable lattices in representations over $p$-adic fields

Amit Ophir (UCSD)

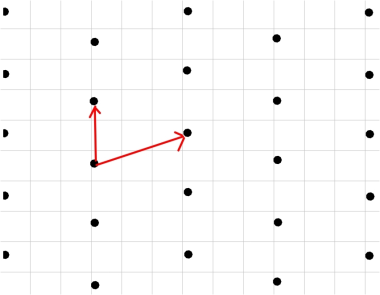
36th Automorphic Forms Workshop

May 23rd, 2024

Joint work with Ariel Weiss (another OSU)

# Warning

A lattice in $\mathbb{R}^2$ is a $\mathbb{Z}$-submodule of rank $2$:



In contrast:
A lattice in $\mathbb{Q}_p^2$ is a $\mathbb{Z}_p$-submodule of rank $2$: **Bounded** and **open**

# Motivation - Ribet's lemma

**Herbrand–Ribet theorem.**

For a prime number $p > 2$:

Divisibility properties of Bernoulli numbers by $p$ $\leftrightsquigarrow$ The structure of the class group of the cyclotomic field $\mathbb{Q}(\mu_p)$

- Generalizations of Ribet's method were used in the proof of...
  1. The Iwasawa main conjecture (over T.R. fields) by Wiles (1990).
  2. The Brumer–Stark conjecture by Dasgupta–Kakde–Silliman–Wang (2024).

# Motivation - Ribet's lemma

- Below: $F = \mathbb{Q}(\mu_p)$.

**Ribet's lemma (1976)**

Suppose that $\rho : G_F \to \mathrm{GL}_2(\mathbb{Q}_p)$ is continuous and irreducible, and

$$\mathrm{trace}(\rho(g)) \equiv \chi_1(g) + \chi_2(g) \pmod{p}$$

for a pair of characters $\chi_1, \chi_2 : G_F \to \mathbb{F}_p^\times$.

# Motivation - Ribet's lemma

- Below: $F = \mathbb{Q}(\mu_p)$.

**Ribet's lemma (1976)**

Suppose that $\rho : G_F \to \mathrm{GL}_2(\mathbb{Q}_p)$ is continuous and irreducible, and

$$\mathrm{trace}(\rho(g)) \equiv \chi_1(g) + \chi_2(g) \pmod{p}$$

for a pair of characters $\chi_1, \chi_2 : G_F \to \mathbb{F}_p^\times$.

There exists a $G_F$-stable lattice $\Lambda \subset \mathbb{Q}_p^2$ such that $\Lambda/p\Lambda$ fits into a **non-split** short exact sequence

$$0 \to \chi_1 \to \Lambda/p\Lambda \to \chi_2 \to 0.$$

# Higher dimensional representations

Let $G$ be a compact group. We fix a continuous **irreducible** representation

$$\rho : G \to \mathrm{GL}_n(\mathbb{Q}_p).$$

# Higher dimensional representations

Let $G$ be a compact group. We fix a continuous **irreducible** representation

$$\rho : G \to \mathrm{GL}_n(\mathbb{Q}_p).$$

- A lattice in $\mathbb{Q}_p^n$ is a $\mathbb{Z}_p$-submodule $\Lambda \subset \mathbb{Q}_p^n$ of rank $n$.

# Higher dimensional representations

Let $G$ be a compact group. We fix a continuous **irreducible** representation

$$\rho : G \to \mathrm{GL}_n(\mathbb{Q}_p).$$

- A lattice in $\mathbb{Q}_p^n$ is a $\mathbb{Z}_p$-submodule $\Lambda \subset \mathbb{Q}_p^n$ of rank $n$.
- A lattice $\Lambda$ is a **stable** if $\rho(g)\Lambda = \Lambda$ for all $g \in G$.

# Higher dimensional representations

Let $G$ be a compact group. We fix a continuous **irreducible** representation

$$\rho : G \to \mathrm{GL}_n(\mathbb{Q}_p).$$

- A lattice in $\mathbb{Q}_p^n$ is a $\mathbb{Z}_p$-submodule $\Lambda \subset \mathbb{Q}_p^n$ of rank $n$.
- A lattice $\Lambda$ is a **stable** if $\rho(g)\Lambda = \Lambda$ for all $g \in G$.
- If $\Lambda$ is a stable lattice, then $\overline{\Lambda} := \Lambda/p\Lambda$ is an $n$-dim. rep. of $G$ ove $\mathbb{F}_p$.

# Higher dimensional representations

Let $G$ be a compact group. We fix a continuous **irreducible** representation

$$\rho : G \to \mathrm{GL}_n(\mathbb{Q}_p).$$

- A lattice in $\mathbb{Q}_p^n$ is a $\mathbb{Z}_p$-submodule $\Lambda \subset \mathbb{Q}_p^n$ of rank $n$.
- A lattice $\Lambda$ is a **stable** if $\rho(g)\Lambda = \Lambda$ for all $g \in G$.
- If $\Lambda$ is a stable lattice, then $\overline{\Lambda} := \Lambda/p\Lambda$ is an $n$-dim. rep. of $G$ ove $\mathbb{F}_p$.
- We denote

$$\overline{\rho}^{ss} = \overline{\Lambda}^{ss} = \bigoplus_{i=1}^{r} W_i^{m_i},$$

for some stable lattice $\Lambda$. The $W_i$ are irreducible and distinct.

# Higher dimensional representations

**Goal (after Ribet)**

Use $\rho$ to construct as many non-split extensions of $W_i$ by $W_j$ as possible.

# Higher dimensional representations

**Goal (after Ribet)**

Use $\rho$ to construct as many non-split extensions of $W_i$ by $W_j$ as possible.

- Say that a stable lattice $\Lambda$ **realizes** a non-split extension of $W_i$ by $W_j$ if there are subrepresentations $U_1 \subset U_2 \subset \overline{\Lambda}$ and a non-split short exact sequence

$$0 \to W_j \to U_2/U_1 \to W_i \to 0.$$

# Higher dimensional representations

**Goal (after Ribet)**

Use $\rho$ to construct as many non-split extensions of $W_i$ by $W_j$ as possible.

- Say that a stable lattice $\Lambda$ **realizes** a non-split extension of $W_i$ by $W_j$ if there are subrepresentations $U_1 \subset U_2 \subset \overline{\Lambda}$ and a non-split short exact sequence

$$0 \to W_j \to U_2/U_1 \to W_i \to 0.$$

- Say that $\rho$ **realizes** a non-split extension of $W_i$ by $W_j$, if some stable lattice $\Lambda$ does.

# Bellaïche's generalization of Ribet's lemma

## Definition

Define the directed graph $\Gamma(\rho)$ as follows.

- Vertices: $W_1, \ldots, W_r$.

# Bellaïche's generalization of Ribet's lemma

## Definition

Define the directed graph $\Gamma(\rho)$ as follows.

- Vertices: $W_1, \dots, W_r$.
- Directed edge from $W_i$ to $W_j$ if $\rho$ realizes a non-split extension of $W_j$ by $W_i$.

# Bellaïche's generalization of Ribet's lemma

## Definition

Define the directed graph $\Gamma(\rho)$ as follows.

- Vertices: $W_1, \ldots, W_r$.
- Directed edge from $W_i$ to $W_j$ if $\rho$ realizes a non-split extension of $W_j$ by $W_i$.

## Theorem (Bellaïche, 2003)

*The graph $\Gamma(\rho)$ is connected as a directed graph.*

# Bellaïche's generalization of Ribet's lemma

## Definition

Define the directed graph $\Gamma(\rho)$ as follows.

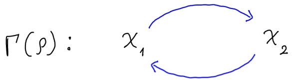
- Vertices: $W_1, \ldots, W_r$.
- Directed edge from $W_i$ to $W_j$ if $\rho$ realizes a non-split extension of $W_j$ by $W_i$.

## Theorem (Bellaïche, 2003)

*The graph $\Gamma(\rho)$ is connected as a directed graph.*

### Example (Original Ribet's lemma)

Suppose that $\rho : G \to \mathrm{GL}_2(\mathbb{Q}_p)$ and $\overline{\rho}^{ss} = \chi_1 \oplus \chi_2$. Then



# Maximal lattices

**Question**

Can we find a small set of stable lattices that capture all of the possible extensions realized by $\rho$?

# Maximal lattices

**Question**

Can we find a small set of stable lattices that capture all of the possible extensions realized by $\rho$?

- Say that a lattice $\Lambda$ is normalized at $0 \neq v \in V$, if $v \in \Lambda$, but $v \notin p\Lambda$.

# Maximal lattices

**Question**

Can we find a small set of stable lattices that capture all of the possible extensions realized by $\rho$?

- Say that a lattice $\Lambda$ is normalized at $0 \neq v \in V$, if $v \in \Lambda$, but $v \notin p\Lambda$.
- Let $\mathcal{L}_v$ be the set of all **stable** lattices normalized at $v$, **ordered by inclusion.**

# Maximal lattices

**Question**

Can we find a small set of stable lattices that capture all of the possible extensions realized by $\rho$?

- Say that a lattice $\Lambda$ is normalized at $0 \neq v \in V$, if $v \in \Lambda$, but $v \notin p\Lambda$.
- Let $\mathcal{L}_v$ be the set of all **stable** lattices normalized at $v$, **ordered by inclusion.**

**Definition**

A stable lattice $\Lambda \subset \mathbb{Q}_p^n$ is **maximal** if it is maximal in some $\mathcal{L}_v$.

# Maximal lattices - algebra

**Theorem (Weiss, O. 2024)**

*Let*

$$0 \to W_i \to U \to W_j \to 0.$$

*be a non-split extension realized by* $\rho$*. Then there exists a maximal lattice* $\Lambda$ *and a subrepresentation of* $\Lambda/p\Lambda$ *isomorphic to* $U$*.*

# Maximal lattices - algebra

**Theorem (Weiss, O. 2024)**

*Let*

$$0 \to W_i \to U \to W_j \to 0.$$

*be a non-split extension realized by $\rho$. Then there exists a maximal lattice $\Lambda$ and a subrepresentation of $\Lambda/p\Lambda$ isomorphic to $U$.*

- Recall that $\overline{\rho}^{ss} = \bigoplus_{i=1}^{r} W_i^{m_i}$.

**Theorem (Weiss, O. 2024)**

*Let $1 \leq i, j \leq r$. Suppose that $m_i = m_j = 1$. Then $\rho$ realizes at most one non-split extension of $W_i$ by $W_j$ (up to automorphisms of $W_i$ and $W_j$).*

# The Bruhat–Tits building

The Bruhat–Tits building (of $\mathrm{PGL}_n(\mathbb{Q}_p)$) $\mathcal{B}_{n-1}$ is the simplicial complex defines as follows.

- Vertices are homothety classes $[\Lambda] = \{a\Lambda \mid a \in \mathbb{Q}_p\}$ of lattices in $\mathbb{Q}_p^n$.
- The vertices $[\Lambda_0], \ldots, [\Lambda_k]$ form a $k$-facet if

$$p\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_k \subsetneq \Lambda_0.$$

# The Bruhat–Tits building

The Bruhat–Tits building (of $\mathrm{PGL}_n(\mathbb{Q}_p)$) $\mathcal{B}_{n-1}$ is the simplicial complex defines as follows.

- Vertices are homothety classes $[\Lambda] = \{a\Lambda \mid a \in \mathbb{Q}_p\}$ of lattices in $\mathbb{Q}_p^n$.
- The vertices $[\Lambda_0], \ldots, [\Lambda_k]$ form a $k$-facet if

$$p\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_k \subsetneq \Lambda_0.$$

- $\mathcal{B}_{n-1}$ has dimension $n-1$ and is 'highly symmetric'.

Denote by $\mathcal{B}_{n-1}(\rho)$ the **invariant subcomplex** of classes of stable lattices.

# The Bruhat–Tits building

The Bruhat–Tits building (of $\mathrm{PGL}_n(\mathbb{Q}_p)$) $\mathcal{B}_{n-1}$ is the simplicial complex defines as follows.

- Vertices are homothety classes $[\Lambda] = \{a\Lambda \mid a \in \mathbb{Q}_p\}$ of lattices in $\mathbb{Q}_p^n$.
- The vertices $[\Lambda_0], \ldots, [\Lambda_k]$ form a $k$-facet if

$$p\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_k \subsetneq \Lambda_0.$$

- $\mathcal{B}_{n-1}$ has dimension $n-1$ and is 'highly symmetric'.

Denote by $\mathcal{B}_{n-1}(\rho)$ the **invariant subcomplex** of classes of stable lattices.

- (Bellaiche, Suh) $\mathcal{B}_{n-1}(\rho)$ is finite if and only if $\rho$ is irreducible.

# The Bruhat–Tits building

The Bruhat–Tits building (of $\mathrm{PGL}_n(\mathbb{Q}_p)$) $\mathcal{B}_{n-1}$ is the simplicial complex defines as follows.

- Vertices are homothety classes $[\Lambda] = \{a\Lambda \mid a \in \mathbb{Q}_p\}$ of lattices in $\mathbb{Q}_p^n$.
- The vertices $[\Lambda_0], \ldots, [\Lambda_k]$ form a $k$-facet if

$$p\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_k \subsetneq \Lambda_0.$$

- $\mathcal{B}_{n-1}$ has dimension $n-1$ and is 'highly symmetric'.

Denote by $\mathcal{B}_{n-1}(\rho)$ the **invariant subcomplex** of classes of stable lattices.

- (Bellaiche, Suh) $\mathcal{B}_{n-1}(\rho)$ is finite if and only if $\rho$ is irreducible.
- $B_{n-1}(\rho)$ is **convex**.

# The Bruhat–Tits building

The Bruhat–Tits building (of $\mathrm{PGL}_n(\mathbb{Q}_p)$) $\mathcal{B}_{n-1}$ is the simplicial complex defines as follows.

- Vertices are homothety classes $[\Lambda] = \{a\Lambda \mid a \in \mathbb{Q}_p\}$ of lattices in $\mathbb{Q}_p^n$.
- The vertices $[\Lambda_0], \ldots, [\Lambda_k]$ form a $k$-facet if

$$p\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_k \subsetneq \Lambda_0.$$

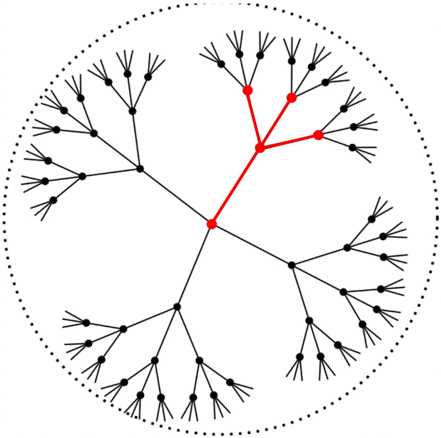
- $\mathcal{B}_{n-1}$ has dimension $n-1$ and is 'highly symmetric'.

Denote by $\mathcal{B}_{n-1}(\rho)$ the **invariant subcomplex** of classes of stable lattices.

- (Bellaiche, Suh) $\mathcal{B}_{n-1}(\rho)$ is finite if and only if $\rho$ is irreducible.
- $B_{n-1}(\rho)$ is **convex**.
- Call a vertex $x = [\Lambda] \in \mathcal{B}_{n-1}(\rho)$ maximal if $\Lambda$ is a maximal lattice.

# Example: $n = 2$

- Let $n = 2$. So $\rho : G \to \mathrm{GL}_2(\mathbb{Q}_p)$.
- $B_1$ is a regular tree. Each vertex has degree $p + 1$.
- Below is $\mathcal{B}_1$ for $p = 3$. In red: the invariant subcomplex $\mathcal{B}_1(\rho)$, for some $\rho$.



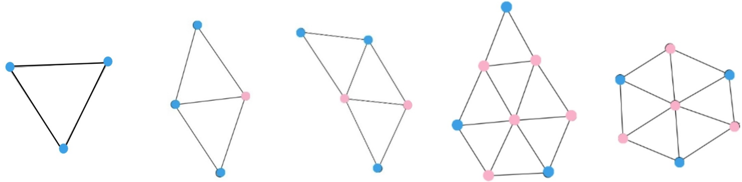
# Example: $n = 3$

- $n = 3$, so $\rho : G \to \mathrm{GL}_3(\mathbb{Q}_p)$.
- Assume that $\overline{\rho}^{ss} = \chi_1 \oplus \chi_2 \oplus \chi_3$ is a sum of three distinct characters.
- $\dim(\mathcal{B}_2(\rho)) = 2$.

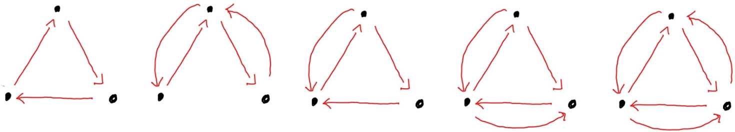
# Example: $n = 3$

- $n = 3$, so $\rho : G \to \mathrm{GL}_3(\mathbb{Q}_p)$.
- Assume that $\overline{\rho}^{ss} = \chi_1 \oplus \chi_2 \oplus \chi_3$ is a sum of three distinct characters.
- $\dim(\mathcal{B}_2(\rho)) = 2$.
- In the pictures below of $\mathcal{B}_{n-1}(\rho)$, the maximal vertices are in blue.

$\mathcal{B}_2(\rho)$ :



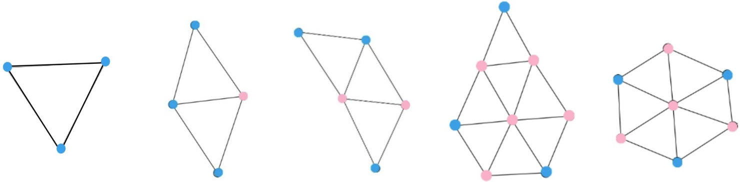
$\Gamma(\rho)$ :



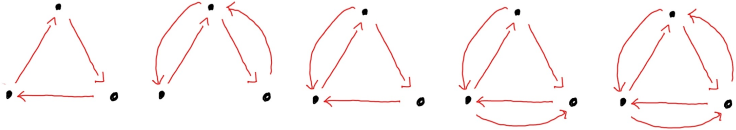
# Example: $n = 3$

- $n = 3$, so $\rho : G \to \mathrm{GL}_3(\mathbb{Q}_p)$.
- Assume that $\overline{\rho}^{ss} = \chi_1 \oplus \chi_2 \oplus \chi_3$ is a sum of three distinct characters.
- $\dim(\mathcal{B}_2(\rho)) = 2$.
- In the pictures below of $\mathcal{B}_{n-1}(\rho)$, the maximal vertices are in blue.

$\mathcal{B}_2(\rho)$ :



$\Gamma(\rho)$ :

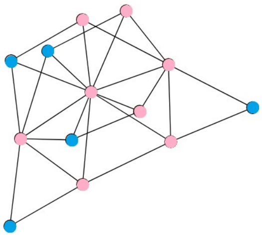


- Bellaïche–Graftieaux: (under some conditions) the shape of $\mathcal{B}_{n-1}(\rho)$ determines $\Gamma(\rho)$ and vice versa.

# Example: $n = 3$

## Example

Let $G = \left\{ \begin{pmatrix} a & * & * \\ 27* & a & * \\ 27* & 9* & a \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}_3) \right\}$. Then $\mathcal{B}_2(G)$ is:



Maximal vertices are in blue.

# Maximal lattices - geometry

- Maximal vertices are always extremal points, and we can use them to improve a lower bound on the number of extremal points of $\mathcal{B}_{n-1}(\rho)$ given by Suh.

**Theorem (A.Weiss, O. 2024)**

$$\dim(\mathcal{B}_{n-1}(\rho)) + 1 \leq |\mathcal{B}_{n-1}^{\max}(\rho)| \leq |\mathcal{B}_{n-1}^{ext}(\rho)|.$$

*If* $m_1 = \cdots = m_r = 1$*, then* $|\mathcal{B}_{n-1}^{\max}(\rho)| = \dim(\mathcal{B}_{n-1}(\rho)) + 1$.

# Maximal lattices - geometry

- Maximal vertices are always extremal points, and we can use them to improve a lower bound on the number of extremal points of $\mathcal{B}_{n-1}(\rho)$ given by Suh.

**Theorem (A.Weiss, O. 2024)**

$$\dim(\mathcal{B}_{n-1}(\rho)) + 1 \leq |\mathcal{B}_{n-1}^{\max}(\rho)| \leq |\mathcal{B}_{n-1}^{ext}(\rho)|.$$

*If* $m_1 = \cdots = m_r = 1$*, then* $|\mathcal{B}_{n-1}^{\max}(\rho)| = \dim(\mathcal{B}_{n-1}(\rho)) + 1$*.*

**Theorem (A.Weiss, O. 2024)**

$\mathcal{B}_{n-1}(\rho)$ *is the tropical convex hull of* $\mathcal{B}_{n-1}^{\max}(\rho)$*.*