Converse Theorem for Jacobi Forms of Half-integral Weight

(Joint work with A. K. Jha and B. Sahu)

Shivansh Pandey (NISER, Bhubaneswar)

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Introduction

- A converse theorem studies the equivalence of Dirichlet series satisfying certain analytic properties and automorphic properties of q-series for certain groups.
- Let $\{c(n)\}$ be a sequence of complex numbers such that $c(n) = O(n^\sigma)$ for some $\sigma > 0$.
- Let $f(\tau) = \sum_{n\geq 1} c(n)e^{(2\pi i n\tau)}$ and $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)\sum_{n\geq 1} c(n)n^{-s}$.

E. Hecke

$f(\tau)$ defines a cusp forms of weight k for $SL_2(\mathbb{Z})$ if and only if $\Lambda(f,s)$ admits analytic continuation to whole complex plane which is bounded on any vertical strip and satisfies

$$\Lambda(f,s) = (-1)^{\frac{k}{2}}\Lambda(f,k-s).$$

- Weil - congruence subgroups.
- Brunier - modular forms of half-integral weights.
- Imai - Siegel modular forms.
- Martin - Jacobi forms of integer index for the full Jacobi group.
- Martin and Osses - congruence subgroups of full Jacobi group.

Jacobi forms of half-integral weights

- $\widetilde{G^J} = \{(\widetilde{\gamma}, X, s) : \gamma \in SL_2(\mathbb{R}),\ X \in \mathbb{R}^2,\ s \in S^1\}$.
- $(\widetilde{\gamma}_1, X, s)(\widetilde{\gamma}_2, Y, s') = \left(\widetilde{\gamma}_1\widetilde{\gamma}_2,\ X\gamma_2 + Y,\ ss' \cdot \det\begin{pmatrix} X\gamma_2 \\ Y \end{pmatrix}\right)$.
- The slash operator $|_{\frac{k}{2},m}$:

$$\begin{aligned}\left(\phi|_{\frac{k}{2},m}h\right)(\tau, z) : \quad &= \quad s^m \varphi(\tau)^{-k} e^m\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + 2\lambda^2\tau + 2\lambda z + \lambda\mu\right) \\ &\times \quad \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right).\end{aligned}$$

- $\Gamma^J(N) = \{(\widetilde{\gamma}, (\lambda, \mu)) : \gamma \in \Gamma_0(N), \lambda \in \mathbb{Z}, \mu \in \mathbb{Z}\}$

Definition

Let χ be a Dirichlet character modulo N. A Jacobi form of weight $\frac{k}{2}$ and index m with character χ for the group $\Gamma^J(N)$ is a complex-valued holomorphic function ϕ defined on $\mathcal{H} \times \mathbb{C}$ satisfying the following conditions:

- $\phi \mid_{\frac{k}{2},m} h = \chi(d)\phi$, for all $h = (\widetilde{\gamma}, X) \in \Gamma^J(N)$ with $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$,
- $\phi \mid_{\frac{k}{2},m} \sigma^{-1} = \sum_{\substack{n,r\in\mathbb{Z} \\ r^2\leq 4nmd_\sigma}} c_{\phi,\sigma}(n,r)e\left(\frac{n}{d_\sigma}\tau + \frac{r}{d_\sigma}z\right)$ for every $\sigma \in SL_2(\mathbb{Q})$.
- Moreover, if the Fourier coefficients $c_{\phi,\sigma}(n,r)$ satisfy $c_{\phi,\sigma}(n,r) = 0$ whenever $4mnd_\sigma = r^2$ for every $\sigma \in SL_2(\mathbb{Q})$, then ϕ is called a Jacobi cusp form.

- $J_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ ($J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$)-space of Jacobi forms (Jacobi cusp forms) of weight $\frac{k}{2}$ and index m with character χ

- $h_\mu(\tau) := \sum_{D=0}^{\infty} c_\mu(D) e_{4m}(D\tau)$.
- Jacobi theta series: $\theta_{l,\mu}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \pmod{2l}}} e\left(\frac{r^2}{4l}\tau + rz\right)$.
- Theta decomposition: $\phi(\tau, z) = \sum_{\mu=1}^{2m} h_\mu(\tau)\theta_{m,\mu}(\tau, z)$.

Definition

$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 < 4nm}} c_\phi(n, r) e(n\tau + rz)$ is called a series of type J, if :

1. The series $\phi(\tau, z)$ converges absolutely and uniformly on every compact subset of $\mathcal{H} \times \mathbb{C}$.
2. $|c_\phi(n, r)| < C(4mn - r^2)^\delta$ for all n, r such that $r^2 < 4nm$.
3. $c_\phi(n, r) = c_\phi(n + \lambda r + \lambda^2 m, r + 2m\lambda)$ for every $\lambda \in \mathbb{Z}$.

The condition (1) and (3) together imply that ϕ has a theta decomposition.

- Let N and M be positive integers with $4|N$, $(N,M)=1$ and χ_1 be a primitive Dirichlet character modulo M.

$$L_\mu(\phi_{\chi_1},s)=\sum_{D=1}^{\infty}\chi_1\left(\frac{D+\mu^2}{4m}\right)c_\mu(D)\left(\frac{D}{4m}\right)^{-s},\ \ \mu\in\{0,1,\cdots,2mM-1\}. \tag{1}$$

- completed Dirichlet series: $\Lambda_\mu(\phi_{\chi_1},s)=\left(\frac{2\pi}{M\sqrt{N}}\right)^{-s}\Gamma(s)L_\mu(\phi_{\chi_1},s)$.

Definition

We call a series $\phi(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z}\\4mn>Nr^2}}c_\phi(n,r)e(n\tau+rNz)$ to be a series of type J_N, if the following properties hold:

1. The series $\phi(\tau,z)$ converges absolutely and uniformly on every compact subset of $\mathcal{H}\times\mathbb{C}$.
2. $|c_\phi(n,r)|<C(4mn-Nr^2)^\delta$ for all n,r such that $Nr^2<4nm$.
3. $c_\phi(n,r)=c_\phi(n+\lambda rN+\lambda^2mN,r+2m\lambda)$ for every $\lambda\in\mathbb{Z}$.

- A series of type J_N has a theta decomposition given by

$$\phi(\tau, z) = \sum_{\mu=1}^{2m} g_\mu(\tau)\theta_{m,\mu}(N\tau, Nz) \tag{2}$$

where $g_\mu(\tau) = \sum_{D=1}^{\infty} d_\mu(D) e\left(\frac{D}{4m}\tau\right)$ and $d_\mu(D) = c_\psi(n, r)$ with $D = 4nm - Nr^2$.

- $L_\mu(\psi_{\chi_1}, s) = \sum_{D=1}^{\infty} \chi_1\left(\frac{D+N\mu^2}{4m}\right) d_\mu(D) \left(\frac{D}{4m}\right)^{-s}$
- $\Lambda_\mu(\psi_{\chi_1}, s) = \left(\frac{2\pi}{M\sqrt{N}}\right)^{-s} \Gamma(s) L_\mu(\psi_{\chi_1}, s).$

Involution operator and twist of Jacobi forms

- Let ϕ be a series of type J or J_N and χ_1 a primitive Dirichlet character modulo M, where $(N, M) = 1$

$$\phi_{\chi_1}(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ r^2\leq 4nm}} \chi_1(n)c_\phi(n, r)e(n\tau + rz). \tag{3}$$

Lemma

Let $\phi \in J_{\frac{k}{2},m}(\Gamma^J(N), \chi)$ be a Jacobi form. Let χ_1 be a primitive Dirichlet character modulo M, where $(N, M) = 1$. Then

$$\phi_{\chi_1}(\tau, z) \in J_{\frac{k}{2},m}(\Gamma^J_{M,1}(NM^2), \chi\chi_1^2).$$

Further, if ϕ is a Jacobi cusp form, then ϕ_{χ_1} is also a Jacobi cusp form.

- For a positive integer L, we define a Fricke involution type operator by

$$W_L(\phi) := (U_{\sqrt{L}}\phi)|_{\frac{k}{2},mL}\widetilde{\gamma}, \tag{4}$$

where $\widetilde{\gamma} = \left(\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, L^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right) \in G$, and the operator U_L is defined as

$$U_L\phi(\tau, Z) := \phi(\tau, Lz).$$

Lemma

Let L be a positive integer with $4|L$ and χ a Dirichlet character modulo L. If $\phi \in J_{\frac{k}{2},m}(\Gamma^J(L), \chi)$, then

$$W_L(\phi) \in J_{\frac{k}{2},mL}(\Gamma^J_{1,L}(L), \chi^*),$$

where $\chi^*(d) = \overline{\chi(d)}\left(\frac{N}{d}\right)$. Further, if ϕ is a Jacobi cusp form, then $W_L(\phi)$ is also a Jacobi cusp form.

Lemma

Let $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ be a Jacobi cusp form. Let χ_1 be a primitive Dirichlet character modulo M, where $(N,M)=1$. Denote $\psi = W_N(\phi)$. Then

$$(W_{NM^2}(\phi_{\chi_1}))(\tau,z) = C_{\chi_1}\psi^*(\tau, Mz),$$

where

$$C_{\chi_1} = \left(\frac{-1}{M}\right)^{\frac{k-1}{2}} \chi(M)\left(\frac{N}{M}\right)\chi_1(-N)\epsilon_M^{-1}\mathcal{G}_{\overline{\chi_1}}^{-1}$$

and

$$\psi^*(\tau,z) = \sum_{u=0}^{M-1} \chi_1(u)\left(\frac{u}{M}\right)\psi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}},(0,0),1).$$

Lemma

Let $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ be a Jacobi cusp form, where χ is a Dirichlet character modulo N. Let M be a prime with $(N,M)=1$. Then

$$B_M(\phi) \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(NM^2),\chi),$$

where $B_M(\phi)$ is defined by $B_M(\phi) := \frac{1}{M}\sum_{u \pmod M} \phi|_{\frac{k}{2},m}(\widetilde{T_{\frac{u}{M}}},(0,0),1)$.

Lemma

For a complex-valued holomorphic function ψ defined on $\mathcal{H}\times\mathbb{C}$, consider the function ψ^* as defined above. Then

- If $\chi_1 \neq \chi_2$ then $C_{\chi_1}\psi^* = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)(\frac{N}{M})\chi_1(-N)\epsilon_M^{-1}\mathcal{G}_{\chi_1\chi_2}\mathcal{G}^{-1}_{\overline{\chi_1}}\psi_{\overline{\chi_1\chi_2}}$.
- If $\chi_1 = \chi_2$, then $C_{\chi_1}\psi^* = (\frac{-1}{M})^{\frac{k-1}{2}}\chi_1(M)(M^{\frac{1}{2}}B_M(\psi) - M^{-\frac{1}{2}}\psi)$.

Here $\chi_2(u) = \left(\frac{u}{M}\right)$.

Proposition[Jha,—, Sahu]

Let k, m and N be positive integers with k odd and $4|N$. Let χ_1 be a character mod M with $(M,N)=1$. If $\phi(\tau,z)$ and $\psi(\tau,z)$ are Fourier series of type J and J_N, respectively. Then the following statements are equivalent:

- There exists a constant C such that

$$(W_{NM^2}(\phi_{\chi_1}))(\tau,z) = C\psi^*(\tau, Mz).$$

- The functions $\Lambda_\mu(\phi_{\chi_1}, s)$ and $\Lambda_\mu(\psi^*, s)$ $(1 \le \mu \le 2mM)$ have a holomorphic continuation to the whole complex plane. Moreover they are bounded in any vertical strip and satisfy the functional equations

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right)\Lambda_\mu(\phi_{\chi_1}, s) = C\Lambda_a\left(\psi^*, \frac{k}{2} - s - \frac{1}{2}\right),$$

  where $1 \le a \le 2mM$.

An immediate corollary to the above proposition is:

Theorem[Jha,—, Sahu]

If $\phi \in J^{cusp}_{\frac{k}{2},m}(\Gamma^J(N),\chi)$ is a Jacobi cusp form with $W_N(\phi)=\psi$, then for each $\mu = 0,1,\cdots,2mM-1$, the completed Dirichlet series $\Lambda_\mu(\phi_{\chi_1},s)$ associated to ϕ admits a holomorphic continuation to the whole complex plane, are bounded in any vertical strip and satisfy the functional equation:

- For $\chi_1 \neq \chi_2(\cdot) = \left(\frac{\cdot}{M}\right)$

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=0}^{2mM-1} e\left(\frac{-a\mu}{2mM}\right)\Lambda_\mu(\phi_{\chi_1},s) = C^{(1)}_{\chi_1}\Lambda_a(\psi_{\overline{\chi_1}\chi_2},\frac{k}{2}-s-\frac{1}{2}),$$

where $C^{(1)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)(\frac{N}{M})\chi_1(-N)\mathcal{G}_{\chi_1\chi_2}\mathcal{G}^{-1}_{\overline{\chi}_1}$.

- For $\chi_1 = \chi_2(\cdot) = \left(\frac{\cdot}{M}\right)$

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=1}^{2mM} e\left(\frac{-a\mu}{2mM}\right)\Lambda_\mu(\phi_{\chi_1},s) = C^{(2)}_{\chi_1}\Lambda_a(M^{\frac{1}{2}}B_M(\psi)-M^{-\frac{1}{2}}\psi,\frac{k}{2}-s-\frac{1}{2})$$

where $C^{(2)}_{\chi_1} = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)$.

Converse Theorem

For a positive integer N, let $\mathcal{M}_N$ be the set of all prime numbers p such that $(p, N) = 1$ and the set $\mathcal{M}_N \cap \{aL + b | L \in \mathbb{Z}\}$ is non-empty for all $a, \ b \in \mathbb{Z}^\times$ with $(a, b) = 1$.

Theorem[Jha,—,Sahu]

Let m, N be positive integers such that $4|N$, and χ be a Dirichlet character modulo N. Let $\{c_\phi(n, r)\}$ and $\{c_\psi(n, r)\}$ be sequences of complex numbers such that the series $\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn>r^2}} c_\phi(n, r)e(n\tau + rz)$ and

$\psi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn>Nr^2}} c_\psi(n, r)e(n\tau + rNz)$ are of type J and J_N, respectively, and

$\psi(\tau, z) = (-1)^{\frac{k}{2}}\overline{\chi}(-1)\psi(\tau, -z)$. Assume that for every primitive Dirichlet character χ_1 of conductor $M \in \mathcal{M} \cup \{1\}$, $\Lambda_\mu(\phi_{\chi_1}, s)$ is entire and bounded in every vertical strip and satisfies the following conditions:

- if $\chi_1 \neq \chi_2 (= (\frac{\cdot}{M}))$, then

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=0}^{2mM-1} e\left(-\frac{a\mu}{2mM}\right)\Lambda_\mu(\phi_{\chi_1}, s) = C^{(1)}_{\chi_1}\Lambda_a(\psi_{\overline{\chi_1}\chi_2}, \frac{k}{2} - s - \frac{1}{2})$$

- if $\chi_1 = \chi_2 = (\frac{\cdot}{M})$, then

$$\left(\frac{2mM}{\sqrt{N}}\right)^{-\frac{1}{2}} \sum_{\mu=0}^{2mM-1} e\left(\frac{-a\mu}{2mM}\right)\Lambda_\mu(\phi_{\chi_1}, s) = C_{\chi_1}^{(2)}\Lambda_a(M^{\frac{1}{2}}B_M(\psi) - M^{-\frac{1}{2}}\psi, \frac{k}{2} - s - \frac{1}{2})$$

where $C_{\chi_1}^{(2)} = (\frac{-1}{M})^{\frac{k-1}{2}}\chi(M)$.
If for every $\mu \in \{0, 1, 2, \cdots, 2mM-1\}$ the Dirichlet series $L_\mu(\phi; s)$ converges absolutely for $\frac{k}{2} - 1 - \epsilon$ for any $\epsilon > 0$, then

$$\phi \in J_{\frac{k}{2},m}^{cusp}(\Gamma^J(N), \chi) \text{ and } \psi = W_N(\phi).$$

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Thank you!