A REDUCIBILITY PROBLEM FOR EVEN UNITARY GROUPS: THE DEPTH ZERO CASE

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NON-ARCHIMEDEAN LOCAL FIELD

- Let F be a field. An absolute value on F is a map $|.| : F \to \mathbb{R}_{\geqslant 0}$ such that for any $x, y \in F$,

$$\begin{gathered} |x| = 0 \iff x = 0, \\ |xy| = |x|\,|y|\,, \\ |x+y| \leqslant |x| + |y|\,. \end{gathered}$$

- We say F is non-Archimedean if

$$|x+y| \leqslant max\{|x|\,, |y|\} \text{ for all } x, y \in F.$$

- The absolute value $|.|$ defines a topology on F which has as a basis for the open sets, all $U(a, \epsilon) = \{b \in F \mid |a-b| < \epsilon\}, a \in F, \epsilon > 0$.

- We call F a non-Archimedean local field if it is locally compact and complete with respect to a non-trivial non-Archimedean absolute value.

- Let

$$\mathfrak{o}_F = \{a \in F \mid |a| \leqslant 1\}$$

which is called the the ring of integers of F. Then $\mathfrak{o}_F$ is a principal ideal domain with unique maximal ideal

$$\mathbf{p}_F = \{a \in F \mid |a| < 1\}.$$

- Let ϖ_F be a generator of the ideal $\mathbf{p}_F$ called a uniformizer of F. We denote $\mathfrak{o}_F/\mathbf{p}_F$ by k_F which is a finite field. We call k_F the residue field of F. We write $|k_F| = q = p^r$ for some prime p and some integer $r \geqslant 1$.
- Note that every element in $x \in F^\times$ can be written uniquely as $x = u\varpi_F^n$, for some unit $u \in \mathfrak{o}_F^\times$ and $n \in \mathbb{Z}$. We use the notation $n = val_F(x)$. In these terms the absolute value on F can be given by $|x| = q^{-val_F(x)} = q^{-n}$ for $x \neq 0$ and $|0| = 0$.

- Let $\mathrm{char}\, F \neq 2$. If $\mathrm{char}\, F = 0$ then F is a finite extension of $\mathbb{Q}_p$ for some prime p.
- Let G be the group of F-points of a connected reductive algebraic group over F. Examples: $G = \mathrm{GL}_n(F), \mathrm{SL}_n(F), \mathrm{Sp}_{2n}(F) \ldots$.
- Via the topology on F, the group G is naturally a locally profinite topological group. Thus G has a neighborhood basis of the identity that consists of compact open subgroups.

Representation of G

- By representation of G (or any locally profinite group), we always mean smooth complex representation (π, V). Precisely,
  - V is a vector space over $\mathbb{C}$ and $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a homomorphism,
  - for each $v \in V$, $\mathrm{Stab}_G(v) = \{g \in G : \pi(g)v = v\}$ is open in G.
- Write $\mathfrak{R}(G) =$ category of representations of G (objects= representations of G, morphisms= G-maps)

Parabolic Induction

- Let P be a parabolic subgroup of G and L be a Levi component of P. Example. For $G = \mathrm{GL}_n(F)$,

$$P = \text{a group of block upper-triangular matrices (up to conjugacy in } G\text{)},$$
$$L = \text{corresponding group of block triangular matrices (up to conjugacy in } P\text{)}.$$

- $P = L \ltimes U$ for a certain subgroup $U =$ the unipotent radical of P.
- Given a representation τ of L, we can view it as a representation of P via the isomorphism $P/U \cong L$ and then induce it to G. This process is called (unnormalized) parabolic induction.
- We need a slight variant where we tensor τ with a certain linear character of L. We write

$$\iota_P^G : \mathfrak{R}(L) \to \mathfrak{R}(G)$$

for the resulting functor of (normalized) parabolic induction. This has better properties and leads to more pleasant formulas. It commutes, for example, with taking duals.

Fundamental Results due to Harish-Chandra and Jacquet

- Let π be an irreducible representation of G.
- By definition, π is cuspidal (or supercuspidal in Harish-Chandra's terminology) if $\mathrm{Hom}_G(\pi, \iota_P^G \tau) = 0$ for all proper parabolic subgroups P of G and all irreducible cuspidal representations τ of L (a Levi component of P). By an easy formal argument,
  - π is cuspidal, or
  - there is a proper parabolic subgroup P of G and an irreducible cuspidal representation τ of L, a Levi component of P, such that π embeds in $\iota_P^G \tau$.

- So the problem of classifying the irreducible representations of G splits naturally into two parts.
  1. Classify the irreducible cuspidal representations of Levi subgroups of G.
  2. Describe how the induced representations $\iota_P^G \tau$, with τ irreducible cuspidal representation of L decompose.

CLASSIFYING THE IRREDUCIBLE REPRESENTATIONS

- Much progress has been made on (I) by constructing irreducible cuspidal representations via induction from suitable compact open subgroups.
- (II) is less well understood. Bushnell and Kutzko have developed a promising approach to (II) via Hecke algebras. To make their method effective, the relevant objects – types and covers and their associated Hecke algebras – must be accessible. One such case is that of depth zero representations: the underlying Hecke algebras, types and covers are well understood by work of Morris.
- We study a special case of (II) in the depth zero setting using Bushnell-Kutzko's approach and Morris results.

Unitary group $\mathrm{U}(n,n)$

- Let $G = \mathrm{U}(n,n)$. More exactly, let E/F be a quadratic Galois extension. Then

$$G = \{g \in \mathrm{GL}_{2n}(E) : {}^{\top}\overline{g} J g = J\}$$

for $J = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$ and $\mathrm{Gal}(E/F) = < - >$.

- P = Standard Siegel parabolic subgroup of G consisting of matrices of the form $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$.
- It has the standard Siegel Levi component

$$L = \{ \begin{bmatrix} a & 0 \\ 0 & {}^{\top}\overline{a}^{-1} \end{bmatrix} : a \in \mathrm{GL}_n(E)\}.$$

- We identify L with $\mathrm{GL}_n(E)$ via the isomorphism

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & {}^{\top}\overline{a}^{-1} \end{bmatrix} : \mathrm{GL}_n(E) \to L.$$

- Write $\mathfrak{o}_E$ for the valuation ring in E and $\mathfrak{p}_E$ for the unique maximal ideal in $\mathfrak{o}_E$.
- We set $k_E = \mathfrak{o}_E/\mathfrak{p}_E$ (the residue field of E).
- Let $K_0 = \mathrm{GL}_n(\mathfrak{o}_E)$. By reducing each matrix entry mod. $\mathfrak{p}_E$, we obtain a surjective homomorphism $r : K_0 \to \mathrm{GL}_n(k_E)$ with kernel
$$K_1 = I_n + \mathrm{M}_n(\mathfrak{p}_E),$$
so that $K_0/K_1 \cong \mathrm{GL}_n(k_E)$.
- An irreducible cuspidal representation τ of $\mathrm{GL}_n(E)$ has depth zero if $\tau^{K_1} \neq 0$ (i.e there exists $v \neq 0 \in V$ where V is the space of τ such that $\tau(k)(v) = v$ for $k \in K_1$).

Types and Hecke Algebras

- Suppose now that τ is an irreducible cuspidal representation of $\mathrm{GL}_n(E)$ of depth zero. The problem is to determine when $\iota_P^G \tau$ is reducible.
- Harish-Chandra tells us to look not at an individual $\iota_P^G \tau$ but at the family $\iota_P^G(\tau\nu)$ as ν varies through the unramified characters of $L \cong \mathrm{GL}_n(E)$.
- The unramified characters of $L \cong \mathrm{GL}_n(E)$ can be defined as the linear characters ν_z for $z \in \mathbb{C}^\times$ given by

$$\nu_z(g) = z^{\mathrm{val}_E(\det g)}, \quad g \in \mathrm{GL}_n(E).$$

- According to Bushnell-Kutzko, via theory of types, we can translate our problem into one about certain Hecke algebra modules.

Types and Hecke Algebras

- Recall $\mathfrak{R}(L)$ denotes the category of representations of L.
- We write $\mathfrak{R}_\tau(L)$ for the full subcategory of $\mathfrak{R}(L)$ whose objects are those representations Σ of L such that each irreducible subquotient of Σ belongs to the family $\tau\nu_z$ ($z \in \mathbb{C}^\times$). In particular, the irreducible objects in $\mathfrak{R}_\tau(L)$ are just the various unramified twists of τ.
- We write $\mathfrak{R}_\tau(G)$ for the full subcategory of $\mathfrak{R}(G)$ consisting of those representations Π of G such that each irreducible subquotient of Π occurs in an induced representation $\iota_P^G(\tau\nu_z)$ for $z \in \mathbb{C}^\times$. Thus the irreducible objects in $\mathfrak{R}_\tau(G)$ are the various irreducible subquotients of the induced representations $\iota_P^G(\tau\nu_z)$ as z varies through $\mathbb{C}^\times$.

- Given a pair (J, ρ) consisting of a compact open subgroup J of G and an irreducible representation ρ of J, we write $\mathfrak{R}_\rho(G)$ for the category of representations of G that are generated by their ρ-isotypic vectors.
- We also write $\mathcal{H}(G, \rho)$ for the intertwining algebra or Hecke algebra determined by ρ. Let $(\rho^\vee, W^\vee)$ be the dual of (ρ, W). Then $\mathcal{H}(G, \rho)$ consists of all compactly supported $\rho^\vee$-spherical functions $\phi : G \to \mathrm{End}_{\mathbb{C}}(W^\vee)$, that is,

$$\phi(j_1 x j_2) = \rho^\vee(j_1)\phi(x)\rho^\vee(j_2), \quad j_i \in J \ (i = 1, 2), \ x \in G.$$

Types and Hecke Algebras

- The categories $\mathfrak{R}_\tau(L)$ and $\mathfrak{R}_\tau(G)$ admit types in the sense of Bushnell-Kutzko. This means that they can be specified by suitable compact open data.
- More exactly, there are pairs (J_L, ρ_L) and (J, ρ) with J_L (resp. J) a compact open subgroup of L (resp. G) and ρ_L (resp. ρ) an irreducible representation of L (resp. G) such that

$$\mathfrak{R}_\tau(L) = \mathfrak{R}_{\rho_L}(L), \;\; \mathfrak{R}_\tau(G) = \mathfrak{R}_\rho(G).$$

- Bushnell-Kutzko's theory then gives explicit equivalences of categories

$$m_L : \mathfrak{R}_\tau(L) \xrightarrow{\simeq} \mathcal{H}(L, \rho_L)\text{-Mod},$$
$$m_G : \mathfrak{R}_\tau(G) \xrightarrow{\simeq} \mathcal{H}(G, \rho)\text{-Mod}.$$

Parabolic induction and Hecke Algebras

- A central feature of Bushnell-Kutzko's theory is that questions about parabolic induction can be translated via these equivalences into questions about appropriate Hecke Algebra modules. This is formalized in their notion of cover.
- In our setting, the type (J, ρ) is a cover of (J_L, ρ_L) (by Morris).
- There is an explicit embedding of $\mathbb{C}$-algebras

$$t = t_P : \mathcal{H}(L, \rho_L) \to \mathcal{H}(G, \rho)$$

Parabolic induction and Hecke Algebras

- The map t gives rise to a functor

$$t_* : \mathcal{H}(L, \rho_L)\text{-Mod} \to \mathcal{H}(G, \rho)\text{-Mod}.$$

- If M is a left $\mathcal{H}(L, \rho_L)$-module, then $t_*(M) = \mathrm{Hom}_{\mathcal{H}(L,\rho_L)}(\mathcal{H}(G, \rho), M)$ where $\mathcal{H}(G, \rho)$ is viewed as a left $\mathcal{H}(L, \rho_L)$-module via t. So the functor t_* looks as below:

$$t_*(M) = \left\{ \psi \colon \mathcal{H}(G, \rho) \to M \;\middle|\; \begin{array}{l} h\psi(h_1) = \psi(t(h)h_1) \text{ where} \\ h \in \mathcal{H}(L, \rho_L), h_1 \in \mathcal{H}(G, \rho) \end{array} \right\}.$$

- The algebra $\mathcal{H}(G, \rho)$ acts by right translations: $\phi.f(\psi) = f(\psi\phi)$ for $f \in \mathrm{Hom}_{\mathcal{H}(L,\rho_L)}(\mathcal{H}(G, \rho), M)$ and $\phi, \psi \in \mathcal{H}(G, \rho)$.

- The functor t_* corresponds to $\iota_P^G : \mathfrak{R}_\tau(L) \to \mathfrak{R}_\tau(G)$ under the equivalences m_L and m_G.

- More precisely, the following diagram commutes (up to a natural equivalence):

$$\begin{array}{ccc} \mathfrak{R}_\tau(G) & \xrightarrow{m_G} & \mathcal{H}(G,\rho)\text{-Mod} \\ \iota_P^G \Big\uparrow & & \Big\uparrow t_* \\ \mathfrak{R}_\tau(L) & \xrightarrow{m_L} & \mathcal{H}(L,\rho_L)\text{-Mod}. \end{array}$$

- Thus

$$\iota_P^G(\tau\nu_z) \text{ is reducible} \iff t_*(m_L(\tau\nu_z)) \text{ is not simple.}$$

Parabolic induction and Hecke Algebras

- The question of whether $t_*(m_L(\tau\nu_z))$ is simple is amenable to direct calculation.
- To carry out this calculation, we compute the structure of the Hecke algebras $\mathcal{H}(L, \rho_L)$ and $\mathcal{H}(G, \rho)$ and identify the embedding t.
- We assume that $\mathrm{char} k_F \neq 2$.

Structure of $\mathcal{H}(G,\rho)$, $\mathcal{H}(L,\rho_L)$ and t

- We set
$$e(E/F) = \frac{2}{[k_E : k_F]} \quad \text{(the ramification index)}.$$
- $\mathcal{H}(G,\rho)$ is generated by two elements ϕ_1, ϕ_2 such that
$$\phi_i^2 = q^{n/e(E/F)} + (q^{n/e(E/F)} - 1)\phi_i \quad (i = 1, 2)$$
with no further relations.
- $$\mathcal{H}(L,\rho_L) \simeq \mathbb{C}[X, X^{-1}],$$
the algebra of complex Laurent polynomials in one variable.
- We may arrange the embedding t such that
$$t(X) = \phi_1\phi_2.$$

Final result

- To state the final result, we need the classification of the irreducible cuspidal representations of $\mathrm{GL}_n(k_E)$. This can be deduced from work of Green or Deligne-Lusztig.
- Let $k = k_E$ and let l/k be a field extension of degree n. We set $\Gamma = \mathrm{Gal}(l/k)$. Let
$$(l^\times)^\vee = \mathrm{Hom}(l^\times, \mathbb{C}^\times).$$
- Clearly, Γ acts on $(l^\times)^\vee$ via
$$\chi^\gamma(x) = \chi({}^\gamma x), \quad \chi \in (l^\times)^\vee,\ \gamma \in \Gamma,\ x \in l^\times.$$

Final result

- We write $(l^\times)^\vee_{\mathrm{reg}}$ for the group of regular characters of $l^\times$ with respect to this action, that is, characters χ such that $\mathrm{Stab}_\Gamma(\chi) = \{1\}$.
- We also write $l^\times_{\mathrm{reg}}$ for the regular elements in $l^\times$, that is, elements x such that $\mathrm{Stab}_\Gamma(x) = \{1\}$.
- The set of Γ-orbits on $(l^\times)^\vee_{\mathrm{reg}}$ is then in canonical bijection with the set $\mathrm{Irr}_{\mathrm{cusp}}\,\mathrm{GL}_n(k)$ consisting of equivalence classes of irreducible cuspidal representations of $\mathrm{GL}_n(k)$:

$$\Gamma\backslash(l^\times)^\vee_{\mathrm{reg}} \longleftrightarrow \mathrm{Irr}_{\mathrm{cusp}}\,\mathrm{GL}_n(k) \qquad \text{(Cusp)}$$
$$\chi \longleftrightarrow \tau_\chi.$$

where the bijection is specified by a character relation:

$$\tau_\chi(x) = c\sum_{\gamma\in\Gamma}\chi^\gamma(x), \quad x \in l^\times_{\mathrm{reg}},$$

for a certain constant c that is independent of χ and x.

Final result

- We can lift (Cusp) to a classification of the irreducible depth zero cuspidal representations of $\mathrm{GL}_n(E)$.
- Let $Z = Z(\mathrm{GL}_n(E))$. Then

$$ZK_0 = \langle \varpi_E \rangle K_0.$$

- Given $\chi \in (l^\times)^\vee_{\mathrm{reg}}$ and $z \in \mathbb{C}^\times$, define a representation $\rho_{\chi,z}$ of ZK_0 by

$$\rho_{\chi,z}(\varpi_E^i k) = z^i \tau_\chi(k), \quad i \in \mathbb{Z},\ k \in K_0,$$

  where we abuse notation slightly in also writing τ_χ for the representation of K_0 inflated from $\mathrm{GL}_n(k)$ via the isomorphism $K_0/K_1 \cong \mathrm{GL}_n(k)$.

Final result

- Write $\mathrm{Irr}^0_{\mathrm{cusp}}\,\mathrm{GL}_n(E)$ for the set of equivalence classes of irreducible cuspidal representations of $\mathrm{GL}_n(E)$ of depth zero.
- Using standard Mackey theory, we obtain a bijection

$$\begin{aligned} \Gamma\backslash(l^\times)^\vee_{\mathrm{reg}} \times \mathbb{C}^\times &\longleftrightarrow \mathrm{Irr}^0_{\mathrm{cusp}}\,\mathrm{GL}_n(E) \\ (\chi, z) &\longleftrightarrow \tau_{\chi,z} := \mathrm{ind}_{ZK_0}^{\mathrm{GL}_n(E)} \rho_{\chi,z}. \end{aligned} \tag{Cusp^0}$$

- The classification of the irreducible cuspidal representations of $\mathrm{GL}_n(E)$ of higher depth is much more complicated.

Getting at the final Theorem

- As our base point in the family of unramified twists of τ, we take $\tau_0 := \tau_{\chi,1}$.
- Note that $m_L(\tau_0\,\nu_z) = \mathbb{C}_z$ where $\mathbb{C}_z$ is the vector space $\mathbb{C}$ where the $\mathcal{H}(L, \rho_L)$ or $\mathbb{C}[X, X^{-1}]$-module structure is given by $X.\lambda = z.\lambda$ for $z \in \mathbb{C}^\times$ and $\lambda \in \mathbb{C}$.
- We have to determine when

$$t_*(m_L(\tau_0\,\nu_z)) = t_*(\mathbb{C}_z)$$

is not simple.

Getting at the final Theorem

- We have to determine when $t_*(\mathbb{C}_z)$ admits a proper subspace that is invariant under the action of ϕ_1 and ϕ_2.
- The final result is

$$\iota_P^G(\tau_0 \nu_z) \text{ is reducible } \iff z^n = -1 \text{ or } q^{\pm n/e(E/F)}.$$

- Looking at central characters, it's easy to see that

$$\tau_{\chi,z}\, \nu_w = \tau_{\chi,zw^n} \quad \text{for any } z, w \in \mathbb{C}^\times.$$

- Hence we have

$$\tau_{\chi,1}\, \nu_{z^{1/n}} = \tau_0\, \nu_{z^{1/n}} = \tau_{\chi,z} \quad \text{for any } z \in \mathbb{C}^\times.$$

Theorem

- E/F is unramified if $[k_E : k_F] = 2$ and ramified otherwise (so $k_E = k_F$).
- We can now state our final result.

Theorem

Assume the residual characteristic of F is odd. We set $q = |k_F|$ and use the notation of (Cusp^0).

a. *For E/F unramified,*

$$\iota_P^G(\tau_{\chi,z}) \text{ is reducible } \iff n \text{ is odd}, \ \chi^{q^{n+1}} = \chi^{-q} \text{ and } z \in \{-1, q^n, q^{-n}\}.$$

b. *For E/F ramified,*

$$\iota_P^G(\tau_{\chi,z}) \text{ is reducible } \iff n \text{ is even}, \ \chi^{q^{n/2}} = \chi^{-1} \text{ and } z \in \{-1, q^{n/2}, q^{-n/2}\}.$$

ODD UNITARY GROUP $\mathrm{U}(n, n+1)$

Theorem

Let $G = \mathrm{U}(n, n+1)$. Let P be the Siegel parabolic subgroup of G and L be the Siegel Levi component of P. Let $\pi = c\text{-}Ind^{L}_{Z(L)\mathfrak{P}_0}\widetilde{\rho_0}$ be a smooth irreducible supercuspidal depth zero representation of $L \cong \mathrm{GL}_n(E) \times \mathrm{U}_1(E)$ where $\widetilde{\rho_0}(\zeta^k j) = \rho_0(j)$ for $j \in \mathfrak{P}_0, k \in \mathbb{Z}$ and $\rho_0 = \tau_\theta$ for some regular character θ of $l^\times$ with $[l : k_E] = n$ and $|k_F| = q$. Consider the family $\iota_P^G(\pi\nu)$ for $\nu \in \mathrm{X}_{nr}(L)$.

1. *For E/F is unramified, $\iota_P^G(\pi\nu)$ is reducible $\Longleftrightarrow$ n is odd, $\theta^{q^{n+1}} = \theta^{-q}$ and $\nu(\zeta) \in \{q^n, q^{-n}, -1\}$.*
2. *For E/F is ramified, $\iota_P^G(\pi\nu)$ is reducible $\Longleftrightarrow$ n is even, $\theta^{q^{n/2}} = \theta^{-1}$ and $\nu(\zeta) \in \{q^{n/2}, q^{-n/2}, -1\}$.*

Future Research

- By the insights of Shahidi, we want to attach a certain L-function called Asai L-function to the smooth irreducible supercuspidal depthzero representation π of the Seigel Levi component $L \cong \mathrm{GL}_n(E)$ of G that we had looked in the previous two projects and compute the reducibility points of the induced representation $\iota_P^G \pi$. Further, note that $\iota_P^G \pi$ is irreducible if and only if the appropriate Asai L-function has a pole at $s = 0$. Further, the L-factors and ϵ-factors for the L-function can also be found out and that is an open ended problem.

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Thank you!