q-series Identities Connected to Ideals in Quadratic Fields

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Outline

1. The q-series σ
2. Application of global class field theory
3. Existing examples of q-series related to quadratic fields
4. Generating novel examples
5. φ_0 as a mock Maass form, and generalizations

The q-series σ

Andrews, Dyson, and Hickerson defined the q-series

$$\sigma(q) := 1 + \sum_{n>0} \frac{q^{n(n+1)/2}}{(-q)_n}.$$

The coefficients of $\sigma(q)$ can be interpreted combinatorially as generated by a partition counting function.

It was conjectured that the coefficients were lacunary, and took arbitrarily large integer values. In fact, they were shown to take every integer value infinitely often.

The q-series σ

The coefficients of $\sigma(q)$ can also be interpreted as counting 'distinct' solutions to Pell's equation

$$x^2 - 6y^2 = m$$

for $m \equiv 1 \pmod{24}$.

The notion of equivalence for solutions (a, b) to Pell's equation corresponds exactly to associates of $a + b\sqrt{6}$ in $\mathbb{Z}[\sqrt{6}]$.

Likewise, the left hand side of Pell's equation is the formula for the norm in $\mathbb{Q}(\sqrt{6})$.

σ as a sum over ideals

With this in mind, a major result of Andrews, Dyson, and Hickerson (1988) shows

$$q\sigma(q^{24}) + q^{-1}\sigma^*(q^{24}) = \sum_{\mathfrak{a}\subseteq\mathbb{Z}[\sqrt{6}]} \chi(\mathfrak{a}) q^{N\mathfrak{a}} \tag{1}$$

where

$$\sigma^*(q) := 2\sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(q;q^2)_n}$$

complements σ for the purposes of this and later results, and χ is a quadratic character on ideals of $\mathbb{Z}[\sqrt{6}]$ relatively prime to $\mathfrak{f} = (12 + 4\sqrt{6})$, and 0 on all other ideals.

Global class field theory summary (à la K. Conrad)

A K-modulus for a number field K is given by

$$\mathfrak{m} = \mathfrak{m}_0 \mathfrak{m}_\infty$$

is where $\mathfrak{m}_0$ is an ideal in $\mathcal{O}_K$, and $\mathfrak{m}_\infty$ is a formal product of real embeddings of K.

We define $I_\mathfrak{m}$ as the group of fractional ideals of $\mathcal{O}_K$ relatively prime to $\mathfrak{m}_0$.

Theorem (Artin, 1927)

For an *admissible* modulus $\mathfrak{m}$ of the extension L/K, there is a surjective group homomorphism

$$\varphi_{L/K,\mathfrak{m}} : I_\mathfrak{m} \to \mathrm{Gal}(L/K).$$

Global class field theory summary (à la K. Conrad)

Define subgroups of $I_{\mathfrak{m}}$:

$N_{\mathfrak{m}}(L/K)$, the group of K fractional ideals which are the norms of fractional ideals in L

$P_{\mathfrak{m}}$, the group of principal ideals (α/β) where

- (α) and (β) are relatively prime to $\mathfrak{m}_0$
- $\alpha \equiv \beta \mod \mathfrak{m}_0$
- $v(\alpha/\beta) > 0$ for $v|\mathfrak{m}_\infty$.

Then

$$A_{\mathfrak{m}}(L/K) := \ker \varphi = N_{\mathfrak{m}}(L/K)P_{\mathfrak{m}},$$

and so the Artin map gives an isomorphism

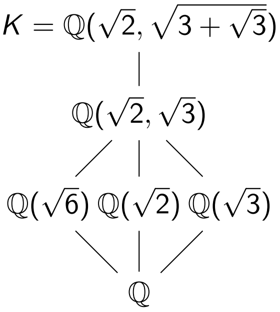
$$I_{\mathfrak{m}}/A_{\mathfrak{m}}(L/K) \simeq \mathrm{Gal}(L/K)$$

where the left-hand side is a subgroup of the ray class group

$$\mathrm{cl}_{\mathfrak{m}}(K) := I_{\mathfrak{m}}/P_{\mathfrak{m}}.$$

Class field theory applied (Cohen)

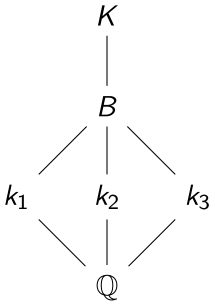
Cohen uses class field theory to explain some of the q-series identities related to σ with the following diagram of field extensions:



- $\mathrm{Gal}(K/\mathbb{Q}) \simeq D_4$
- Two characters χ_i, χ_i' on each $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{D}))$ induce the unique irreducible representation of degree 2, ρ of $\mathrm{Gal}(K/\mathbb{Q})$
- $L(\rho, s) = L(\chi_i, s)$

$$\sum_{\mathfrak{a} \subset \mathbb{Z}[\sqrt{6}]} \chi_1(\mathfrak{a}) q^{N\mathfrak{a}} = \sum_{\mathfrak{a} \subset \mathbb{Z}[\sqrt{2}]} \chi_2(\mathfrak{a}) q^{N\mathfrak{a}} = \sum_{\mathfrak{a} \subset \mathbb{Z}[\sqrt{3}]} \chi_3(\mathfrak{a}) q^{N\mathfrak{a}}$$

Class field theory applied (Cohen)



- Let K be a Galois extension of $\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q})$ has a cyclic center Z of index 4.
- $K^Z = B$ is a biquadratic subfield.
- For each quadratic subfield, k_i, two characters on $\mathrm{Gal}(K/k_i)$ induce the unique irreducible representation of degree 2.

$$\sum_{\mathfrak{a}\subset\mathcal{O}_{k_1}} \chi_1(\mathfrak{a})q^{N\mathfrak{a}} = \sum_{\mathfrak{a}\subset\mathcal{O}_{k_2}} \chi_2(\mathfrak{a})q^{N\mathfrak{a}} = \sum_{\mathfrak{a}\subset\mathcal{O}_{k_3}} \chi_3(\mathfrak{a})q^{N\mathfrak{a}}$$

Other q-series identities

Further identities have been given by the following:

- Corson, Favero, Liesenger, and Zubairy (2004): a pair of q-series related to $\mathbb{Q}(\sqrt{2})$.

$$W_1(q) := \sum_{n \geq 0} \frac{(q;q)_n (-1)^n q^{\binom{n+1}{2}}}{(-q;q)_n}$$

$$W_2(q) := \sum_{n \geq 1} \frac{(-1;q^2)_n (-q)^n}{(q;q^2)_n}.$$

These yield an analogous identity to (1):

$$qW_1(-q^8) + q^{-1} W_2(-q^8) = \sum_{\substack{\mathfrak{a} \subset O_K \\ N(\mathfrak{a}) \equiv 1 \pmod{2}}} q^{N(\mathfrak{a})}.$$

Other q-series identities

Further identities have been given by the following:

- Corson, Favero, Liesenger, and Zubairy (2004): a pair of q-series related to $\mathbb{Q}(\sqrt{2})$.
- Lovejoy (2004): three identities relating overpartition generating functions to ideals in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, and $\mathbb{Q}(\sqrt{6})$.
- Bringmann and Kane (2011): eight total identities related to $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$.
- Lovejoy and Osburn (2015): twelve total identities related to $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, and $\mathbb{Q}(\sqrt{6})$.

How I hope to build on this work

Notice that all the existing identities relate q-series to sums over ideals in the integer rings of $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, or $\mathbb{Q}(\sqrt{6})$.

Cohen's theory implies the existence of relations between sums over ideals in rings of integers for any two quadratic fields.

- He gives one additional explicit example with fields $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(i)$, and the Dedekind eta function.
- I have not found other explicit examples in the literature.

Generating similar series

My goal is to generate starting terms of the q-series described by Cohen for different quadratic fields

- The LMFDB (L-functions and modular forms database) contains (10s of thousands of) examples of degree 8 number fields whose Galois group is D_4 or Q_8.
  - Any of these will satisfy Cohen's hypotheses.
- Pari/GP has built in functions to compute:
  - the conductor $\mathfrak{f}$ (a particular admissable modulus) for a given field extension
  - the corresponding subgroup of $\mathrm{cl}_{\mathfrak{f}}$
  - the structure of $\mathrm{cl}_{\mathfrak{f}}$, and in particular its generators
  - generate all ideals of a particular norm.
- With this computed it suffices, for each ideal $\mathfrak{a}$:
  - to check that $\mathfrak{a}$ is relatively prime to $\mathfrak{f}$
  - to check which class of $\mathrm{cl}_{\mathfrak{f}}$ (coset of $P_{\mathfrak{f}}$) contains $\mathfrak{a}$
  - to count $\mathfrak{a}$ in the coefficient of $q^{N\mathfrak{a}}$ with the root of unity determined by χ.

Checking which coset of $P_{\mathfrak{f}}$ an ideal $\mathfrak{a}$ falls into

If we want to check whether $\mathfrak{a}$ falls in $\mathfrak{b}P_{\mathfrak{f}}$, we equivalently check if $\mathfrak{a}\mathfrak{b}^{-1} \in P_{\mathfrak{f}}$.

If we additionally know that the quadratic field has class number 1, then we can assume the all the ideals involved are principal. ($\mathfrak{a} = (\alpha)$, $\mathfrak{b} = (\beta)$, $\mathfrak{f}_0 = (f)$)

Then we only need to check that

- $u\alpha - \beta$ is a multiple of f for some unit u, and
- $v(\alpha/\beta) > 0$ for $v|\mathfrak{f}_\infty$.

Checking which coset of $P_{\mathfrak{f}}$ an ideal $\mathfrak{a}$ falls into

I am currently in the process of writing and debugging scripts to do this, starting by verifying the examples from Andrews, Dyson, Hickerson, and Cohen.

My next step will be to identify novel q-series and investigate if they possess any combinatorial or analytic properties analogous to earlier examples.

Defining φ_0

In investigating analytic and modular properties of forms related to σ, Cohen first looks at

$$\varphi(q) := q^{1/24}\sigma(q) + q^{-1/24}\sigma^*(q) = \sum_{n\in\mathbb{Z}} T(n)q^{|n|/24}.$$

Recall (1),

$$q\sigma(q^{24}) + q^{-1}\sigma^*(q^{24}) = \sum_{\mathfrak{a}\subseteq\mathbb{Z}[\sqrt{6}]} \chi(\mathfrak{a})q^{N\mathfrak{a}}.$$

He then defines the related form ($\tau = x + iy$)

$$\varphi_0(\tau) := y^{1/2} \sum_{n\in\mathbb{Z}^*} T(n)e^{2i\pi nx/24}K_0(2\pi|n|y/24)$$

where K_0 is a modified Bessel function of the second kind.

Properties of φ_0

Theorem (Cohen, 1988)

(i) φ_0 is an eigenfunction of the hyperbolic Laplace operator $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ with eigenvalue $\lambda = 1/4$.

(ii) $\varphi_0\left(\frac{-1}{2\tau}\right) = \overline{\varphi_0(\tau)}$.

(iii) For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ we have $\varphi_0\left(\frac{a\tau+b}{c\tau+d}\right) = v(\gamma)\varphi_0(\tau)$, where $|v(\gamma)| = 1$ and v is a multiplier system defined uniquely by

$$v\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = v\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = e^{2i\pi/24}.$$

Generalizations of φ_0

Further investigations have placed φ_0 in a more general framework and uncovered similar results for some of the series discovered by other authors:

- Zwegers (2011): Developed a generalization of φ_0, which he termed mock Maass theta functions, which yield actual Maass waveforms in special cases.
- Li, Ngo, and Rhoades (2013): Developed a process termed "renormalization" to recover the remaining Fourier coefficients of a Maass waveform when half are known.
- Krauel, Rolen, and Woodbury (2016): Developed an alternative approach and were able to explicitly place many of the examples from Bringmann and Kane (2011), and Lovejoy (2004) within this framework.
- Li and Roehrig (2023): Using a different method involving lattice algebra and building on other work by Funke and Kudla (2017) and Charollois and Li (2020), re-proved the results of Zwegers, and constructed new modular forms.

Further Directions

Once I am able to use my program to generate novel examples of q-series stemming from characters on ideals in integer rings of quadratic fields, I hope to be able to use resources like the LMFDB, and recent work in this area to find such series with:

- connections to known hypergeometric series, modular forms, and/or L-functions
- interesting combinatorial interpretations
- coefficients yielding novel Maass waveforms or other families of modular forms

Thank you for listening!