On real zeros of the Hurwitz zeta function

Karin Ikeda（Kyushu University D1）

36th Automorphic Forms Workshop

Outline

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1. Introduction
2. Previous works (Real zero)
3. Main result

Introduction

About the Hurwitz zeta function

Definition (Hurwitz zeta function)

For $s = \sigma + it$, $\sigma > 1$, $0 < a \leq 1$,

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

Remark)

- The series converges absolutely and locally uniformly in the half-plane $\sigma > 1$.
- $\zeta(s, a)$ is analytically continued to the whole complex plane except for a simple pole at $s = 1$.

Introduction

Example)

- If $a = 1$

$$\zeta(s,1) = \zeta(s) \text{ (Riemann zeta).}$$

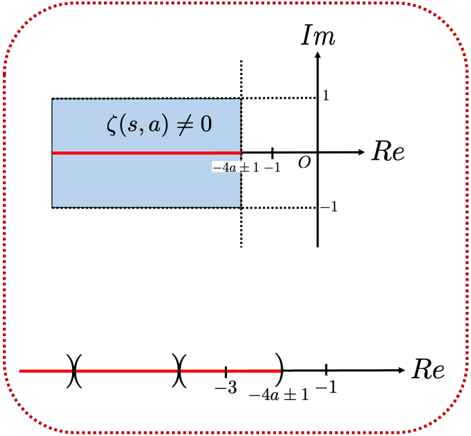
- If $a = \frac{1}{2}$,

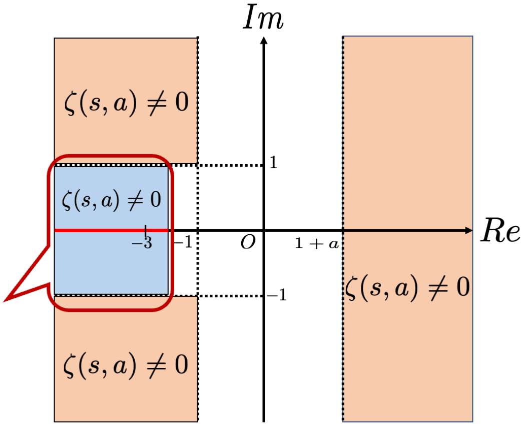
$$\zeta\left(s, \frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^s} = 2^s \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}$$

$$= 2^s \left(\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \right) = (2^s - 1)\zeta(s).$$

Introduction

Zero-free region of $\zeta(s,a)$. (Spira '76)





Outline

Previous works (Real zero)

Existence of real zeros

Classical

For $n \geq 0$ ($n \in \mathbb{N}$) we have

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1},$$

where $B_n(x)$ is the nth Bernoulli polynomial.

Example)

$$B_0(x) = 1 \qquad B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6} \qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

Previous works (Real zero)

Existence of real zeros

Theorem (Matsusaka, 2018)

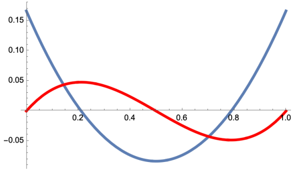
Let $N \geq 0$ be an integer.

$$\zeta(\sigma, a) \text{ has a zero in the interval } (-N, -N+1). \iff B_N(a)B_{N+1}(a) < 0.$$

Note that the Hurwitz zeta function has no real zero in $\sigma > 1$

- For example $N = 1$

$$B_2(a)B_3(a) < 0$$
$$\Leftrightarrow 0.2113\cdots < a < 0.5,\ 0.7886\cdots < a < 1.$$

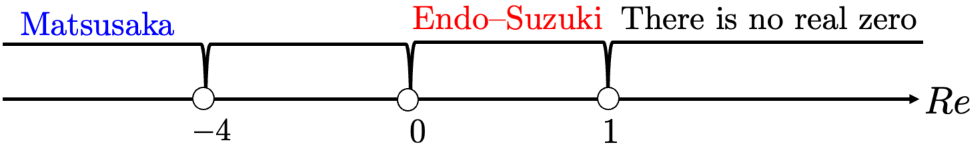


Previous works (Real zero)

Uniqueness of real zeros

Theorem (Endo–Suzuki, 2019, Matsusaka, 2018)

Suppose $N \geq 5$ or $N = 0$. If $B_N(a)B_{N+1}(a) < 0$, the zero of $\zeta(\sigma, a)$ in the interval $(-N, -N+1)$ is unique and simple.

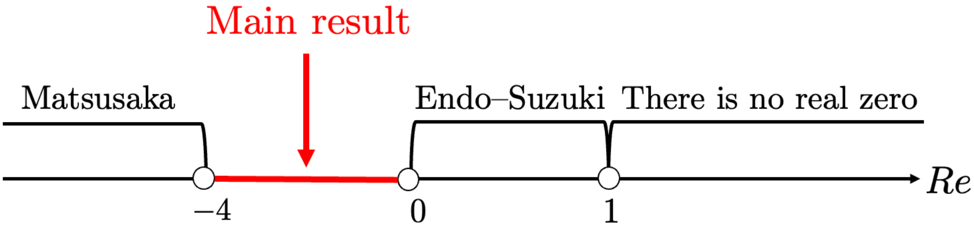


Outline

Main result

Theorem (I., 2023)

For $1 \leq N \leq 4$, the zero of $\zeta(\sigma, a)$ in the interval $(-N, -N+1)$ is also unique and simple If $B_N(a)B_{N+1}(a) < 0$.



Main result

Sketch of proof

Let
$$H_N(a,x) := \frac{e^{(1-a)x}}{e^x - 1} - \sum_{n=0}^{N} \frac{B_n(1-a)}{n!} x^{n-1},$$
$$h_N(a,x) := x(e^x - 1)H_N(a,x),$$
$$f_N(a,x) := e^{(a-1)x} \frac{\partial^{N+1}}{\partial x^{N+1}} h_N(a,x).$$

Then, $\frac{\partial}{\partial x} f_N(a,x)$ has exactly one zero in $x > 0$.

$\Longrightarrow \zeta(\sigma,a)$ has a unique simple real zero in $(-N, -N+1)$.

Main result

We will show

$$\text{“}\underbrace{e^{-ax}\frac{\partial^2}{\partial x^2}f_N(a,x)}_{\text{degree } N} \text{ has either no or one real zero in } x>0.\text{”}$$

Then, $\frac{\partial}{\partial x}f_N(a,x)$ has exactly one zero in $x>0$.

Example)

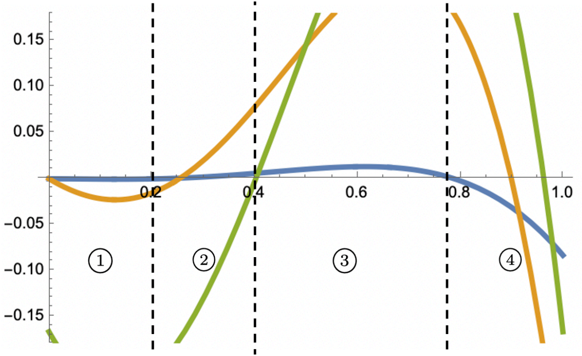
- $N=1$

$e^{-ax}\frac{\partial^2}{\partial x^2}f_1(a,x)$ is linear polynomial. So, this case is trivial.

Main result

$$e^{-ax}\frac{\partial^2}{\partial x^2}f_2(a,x)=0$$

$$\Leftrightarrow \underbrace{-a^2 6B_2(a)}_{C_2(a)}x^2 \underbrace{-4a(9a^3-6a^2-3a+1)}_{C_1(a)}x \underbrace{-2(18a^4-24a^2+6a+1)}_{C_0(a)}=0$$



①, ③ : $\alpha\beta > 0, \alpha+\beta < 0$

$\rightarrow$ negative solution : 2

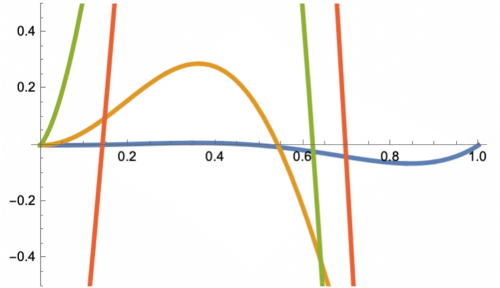
②, ④ : $\alpha\beta < 0$

$\rightarrow$ positive solution : 1

negative solution : 1

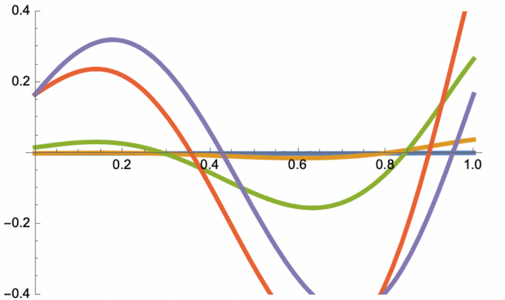
Main result

- $N = 3$ same as $N = 2$



$$e^{-ax}\frac{\partial^2}{\partial x^2}f_3(a,x) = C_{3,3}(a)x^3 + C_{3,2}(a)x^2 + C_{3,1}(a)x + C_{3,0}(a)$$

- $N = 4$ same $+\ \alpha$



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Main result

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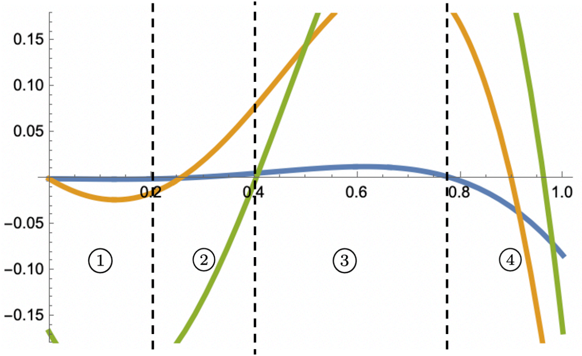

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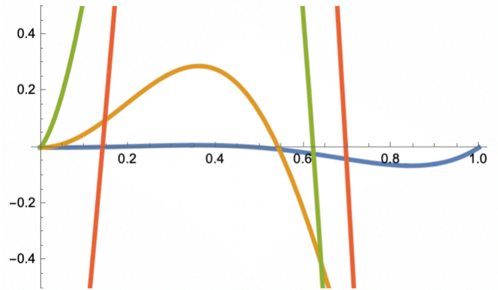

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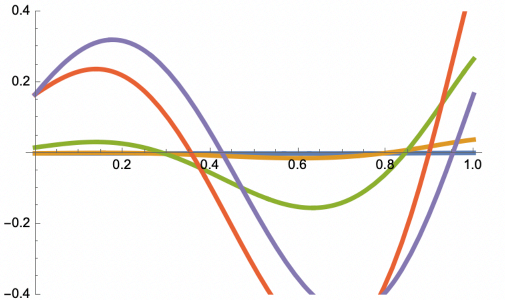

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Thank you for your attention