Improvements to Sturm Bounds

Peter Marcus

Tulane University

Definitions

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Question

How well can we bound $m(N,k,\nu)$?

Motivations

Congruences

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Weierstrass Points

∞ is not a Weierstrass point of $X_0(N)$ iff $m(N, 2, \infty) = d(N, 2)$.

First observations

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Observation 2

$$m(N, k, \nu) \geq d(N, k)$$

Miller Basis

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A **Miller basis** is a basis of $S_k(N)$ of the form

$$\begin{array}{rccccc} \mathrm{f}_1 = & \mathrm{q}+ & \cdots & \cdots & +O(\mathrm{q}^{d+1}) \\ \mathrm{f}_2 = & & \mathrm{q}^2+ & \cdots & +O(\mathrm{q}^{d+1}) \\ \vdots & & \ddots & & \\ \mathrm{f}_d = & & & \mathrm{q}^d & +O(\mathrm{q}^{d+1}) \end{array}$$

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Easy Facts

Iff $m(N,k,\infty) = d(N,k)$, then $S_k(N)$ has a Miller basis.
Iff $m(N,k,p) = d(N,k)$, then $S_k(N)$ has a p-integral Miller basis.
Iff $m(N,k,\nu) = d(N,k)$ for all ν, then $S_k(N)$ has an integral Miller basis.

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Propostion (Miller)

$S_k(1)$ has an integral Miller basis.

Sturm's Bound

Theorem (Sturm '87)

Let $r = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$. Then

$$m(N, k, \nu) \leq m(1, kr, \nu) \leq \frac{kr}{12}$$

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Theorem

If $p \mid N$, then

$$m(pN, k, \nu) \leq m(N, kp, \nu)$$

If $p \nmid N$, then

$$m(pN, k, \nu) \leq m(N, k(p+1), \nu)$$

Alghren—Masri—Rouse Improvements

Theorem 1 (Alghren—Masri—Rouse '09)

$$m(N, k, \nu) \leq \frac{kr}{12} - \left\{\frac{k}{3}\right\}\epsilon_3(N) - \left\{\frac{k}{4}\right\}\epsilon_2(N) - \epsilon_\infty(N) + 1$$

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$$= d(N,k) + g(N) - \delta_{k,2}$$

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Theorem 2 (Alghren—Masri—Rouse '09)

If $p \geq 5$ and $p > k$, then

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Note: reduces $k(p+1)$ to kp in the $p \nmid N$ case.

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Corollary (— '24)

With the same conditions on p, and if the genus of $X_0(N)$ is 0, then

$$m(pN, k, \infty) \leq m(N, kp, \infty)$$

Wang Improvements

Theorem (Wang '23)

If $p \geq 5$, $p > k$, $p \nmid N$, and $d(N, k) = 0$, then

$$m(pN, k, p) = m(pN, k, \infty) = d(pN, k)$$

Wang's Weight Periodicity

Theorem (Wang '23)

Define

$$S_k(N)^n := \{f \in S_k(N) : \mathrm{ord}_\nu(f) > n\}$$

There is an explicit constant $w(N)$ such that

$$S_k(N)^n \cong S_{k+w(N)}(N)^{n+rw(N)/12}$$

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Corollary (— '24)

As long as $m(N, k, \nu) > 0$,

$$m(N, k + w(N), \nu) = m(N, k, \nu) + \frac{rw(N)}{12}$$

Explicit Values of $w(N)$

N	$w(N)$
1	12
2	4
3	6
2^n	2
$2^m p^n$	$p-1$
otherwise	$(\prod_{p\mid N}(p-1))/2$

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Obtains sharp bound for $m(N, k, \infty)$ for $N \leq 20$.
$m(21, 6, \infty) = 12$, but the best bound gives $m(21, 6, \infty) \leq m(3, 42, \infty) = 13$.

Examples

$S_4(11)$

- $m(11, 4) = 2$
- Sturm: $m(11, 4) \leq 4$
- AMR: $m(11, 4) \leq 3$
- Wang: $m(11, 4) \leq 2$

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- Wang: $m(11, 4) \leq 2$

$S_6(21)$

- $m(21, 6) = 12$
- Sturm: $m(21, 6) \leq 16$
- AMR/Wang: $m(21, 6) \leq 13$

A lower bound

Proposition

$$m(pN, k, \nu) \geq pm(N, k, \nu)$$

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Corollary

Suppose $p \geq 5, p > k$, and $12|k$. Then

$$m(p, k, \infty) = \frac{kp}{12}$$

Future Directions

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- $M_k(\Gamma_0(N))$

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- $M_k(\Gamma_0(N))$
- $M_k(N, \chi)$
- etc...
- Vanishing at other cusps/other points

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