Distribution of rationality fields

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(joint with Alex Cowan)

Level 1

$f = \sum a_n q^n \in S_k(1)$ - newform (normalized eigenform)
$K_f = \mathbb{Q}(\{a_n\}_n)$ - rationality field (number field)
Galois action on newforms $\implies [K_f : \mathbb{Q}] \leq \dim S_k(1)$

Examples

$k = 12$: $q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \dots$
$k = 16, 18, 20, 22, 26$ - similar

$k = 24$:
$q + (540 - \alpha)q^2 + (169740 + 48\alpha)q^3 + (12663328 - 1080\alpha)q^4 + \dots$
$\alpha = 12\sqrt{144169}$

Conjecture (Maeda's conjecture, 1997)

All newforms in $S_k(1)$ are Galois conjugate. Equivalently, every rationality field has degree $d = \dim S_k(1)$.

Fixed level

$N = p_1 \dots p_m$ - squarefree (for simplicity)
$f \in S_k^{\mathrm{new}}(N)$ - newform

Typically $[K_f : \mathbb{Q}] < \dim S_k^{\mathrm{new}}(N)$. Why?

Atkin–Lehner operators $W_{p_1}, \dots, W_{p_m}$ give Galois-stable decomposition

$$S_k^{\mathrm{new}}(N) = \bigoplus_{\varepsilon} S_k^{\mathrm{new}}(N)^{\varepsilon}$$

$\varepsilon = (\varepsilon_{p_1}, \dots, \varepsilon_{p_m})$ - sign pattern ($\varepsilon_{p_i} = \pm 1$)
$S_k^{\mathrm{new}}(N)^{\varepsilon}$ – Atkin-Lehner eigenspace (joint kernel of each $W_{p_i} - \varepsilon_{p_i}$)

Conjecture (Tsaknias' generalized Maeda conjecture, 2014)

For $k \gg_N 0$, all newforms in each $S_k^{\mathrm{new}}(N)^{\varepsilon}$ are Galois conjugate.

Weight 2 newforms on the LMFDB

Label	Dim	A	Field	Traces				Fricke sign	q-expansion
				a_2	a_3	a_5	a_7		
11.2.a.a	1	0.088	$\mathbb{Q}$	-2	-1	1	-2	$-$	$q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \cdots$
14.2.a.a	1	0.112	$\mathbb{Q}$	-1	-2	0	1	$-$	$q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + \cdots$
15.2.a.a	1	0.120	$\mathbb{Q}$	-1	-1	1	0	$-$	$q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + \cdots$
17.2.a.a	1	0.136	$\mathbb{Q}$	-1	0	-2	4	$-$	$q - q^2 - q^4 - 2q^5 + 4q^7 + 3q^8 - 3q^9 + \cdots$
19.2.a.a	1	0.152	$\mathbb{Q}$	0	-2	3	-1	$-$	$q - 2q^3 - 2q^4 + 3q^5 - q^7 + q^9 + 3q^{11} + \cdots$
20.2.a.a	1	0.160	$\mathbb{Q}$	0	-2	-1	2	$-$	$q - 2q^3 - q^5 + 2q^7 + q^9 + 2q^{13} + \cdots$
21.2.a.a	1	0.168	$\mathbb{Q}$	-1	1	-2	-1	$-$	$q - q^2 + q^3 - q^4 - 2q^5 - q^6 - q^7 + \cdots$
23.2.a.a	2	0.184	$\mathbb{Q}(\sqrt{5})$	-1	0	-2	2	$-$	$q - \beta q^2 + (-1+2\beta)q^3 + (-1+\beta)q^4 + \cdots$
24.2.a.a	1	0.192	$\mathbb{Q}$	0	-1	-2	0	$-$	$q - q^3 - 2q^5 + q^9 + 4q^{11} - 2q^{13} + \cdots$
26.2.a.a	1	0.208	$\mathbb{Q}$	-1	1	-3	-1	$-$	$q - q^2 + q^3 + q^4 - 3q^5 - q^6 - q^7 + \cdots$
26.2.a.b	1	0.208	$\mathbb{Q}$	1	-3	-1	1	$-$	$q + q^2 - 3q^3 + q^4 - q^5 - 3q^6 + q^7 + \cdots$
27.2.a.a	1	0.216	$\mathbb{Q}$	0	0	0	-1	$-$	$q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots$
29.2.a.a	2	0.232	$\mathbb{Q}(\sqrt{2})$	-2	2	-2	0	$-$	$q + (-1+\beta)q^2 + (1-\beta)q^3 + (1-2\beta)q^4 +$

rational ($K_f = \mathbb{Q}$) newforms $\longleftrightarrow$ elliptic curves

$\implies$ $K_f = \mathbb{Q}$ occurs infinitely often (but should be 0% of the time)

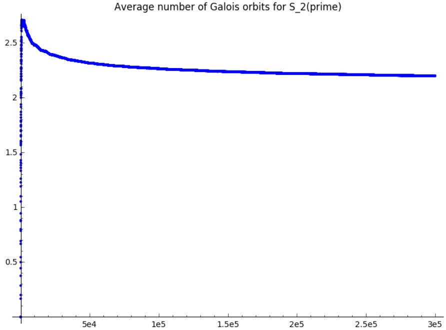
Typical decomposition for fixed weight

Conjecture (Lipnowski–Shaeffer, 2020)

As (squarefree) $N \to \infty$, $\max\{[K_f : \mathbb{Q}]\} \sim \dim S_k^{\mathrm{new}}(N)^{\varepsilon}$.

Conjecture 1 (M., 2021)

As (squarefree) $N \to \infty$, *on average (in particular, 100% of the time) each Atkin–Lehner eigenspace has a single Galois orbit.*



Counting rationality field degrees in weight 2

Conjecture (Brumer–McGuinness, Watkins)

The number of elliptic curves with (squarefree or arbitrary) conductor $N < X$ *grows like* $cX^{5/6}$.

Conjecture 2 (Cowan–M)

Fix d *and restrict to squarefree levels* N. *Then*

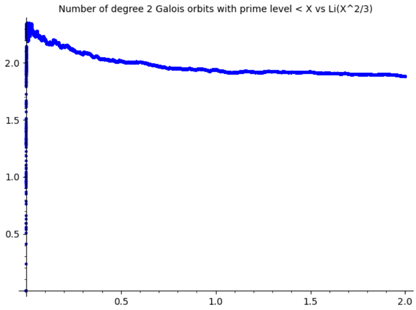
$$\#\{\textit{newforms } f \in S_2(N) : N < X, [K_f : \mathbb{Q}] = d\} = O(X^{1-d/6+\varepsilon}).$$

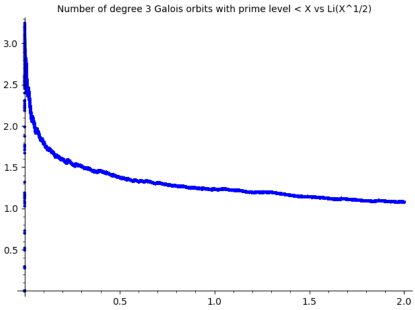
In particular, this is count is finite if $d > 6$.

— conjectural *upper* bound based on random Hecke polynomial model and data for prime $N < 2,000,000$

— actual count should be infinite if $d = 2, 3$, probably $d = 4$; not clear for $d = 5, 6$

Comparisons to upper bounds for prime level





Prime level versions ($N < 2.0 \times 10^6$) of:

$d = 2$ count divided by $X^{2/3}$ $d = 3$ count divided by $X^{1/2}$

Counting specific rationality fields

Shimura $+\varepsilon$

$$\{\text{newforms } f \in S_2(N) : K_f = K\} \longleftrightarrow$$
$$\left\{\text{abelian varieties } A : d = \dim A = [K:\mathbb{Q}],\ N_A = N^d,\ \mathrm{End}^0_{\mathbb{Q}}(A) = K\right\}$$

$\mathrm{End}^0_{\mathbb{Q}}(A) = K$ means A has **real multiplication** (**RM**) by K (or $\mathcal{O} \subset K$)

Moduli spaces

Modular curve $X_0(1) \sim \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$

$$\{\text{ell. curves}/\mathbb{Q}\} \longleftrightarrow X_0(1)(\mathbb{Q})$$

Hilbert modular varieties $Y_{d,K,\varepsilon}$

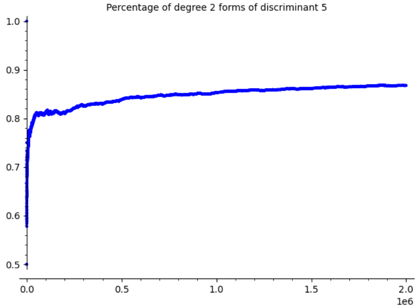
$$\{d\text{-dim AVs } A/\mathbb{Q} \text{ with RM by } K+\varepsilon\} \longrightarrow Y_{d,K,\varepsilon}(\mathbb{Q})$$

The most common quadratic rationality field

Counting rational points on Hilbert modular surfaces ($d = 2$) $\rightsquigarrow$

Conjecture 3 (Cowan–M)

Among weight 2 newforms with $[K_f : \mathbb{Q}] = 2$, 100% have $K_f = \mathbb{Q}(\sqrt{5})$.



Lower bounds for quadratic fields

Using constructions of genus 2 curves with RM $\mathbb{Q}(\sqrt{5})$ (by Brumer) and RM $\mathbb{Q}(\sqrt{2})$ (by Mestre) $\rightsquigarrow$

Proposition 4 (Cowan–M)

If we do not restrict to squarefree N, we have the lower bounds

$$\#\left\{\textit{new, min } f \in S_2(N) : N < X, K_f = \mathbb{Q}(\sqrt{5})\right\} \gg X^{1/3}$$

$$\#\left\{\textit{new, min } f \in S_2(N) : N < X, K_f = \mathbb{Q}(\sqrt{2})\right\} \gg X^{2/7}$$

Cowan–Frengley–M: constructions of genus 2 curves with RM $\mathbb{Q}(\sqrt{D})$ for $D = 5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 44, 53, 61$

$\overset{?}{\Longrightarrow}$ lower bounds for such $\mathbb{Q}(\sqrt{D})$

What quadratic fields arise

Question

What quadratic fields occur as rationality fields of weight k newforms?

Conjecture 5

Only finitely many quadratic fields occur (for all k).

LMFDB searches for maximal discriminants:

$k = 2$: $\mathbb{Q}(\sqrt{145})$ $(N = 3300)$
$k = 4$: $\mathbb{Q}(\sqrt{8761})$ $(N = 1050)$
$k = 6$: $\mathbb{Q}(\sqrt{176089})$ $(N = 210)$
...
$k = 60$: $\mathbb{Q}(\sqrt{6597958871807685154735396 81})$ $(N = 6)$