Explicit Linear Relations between Special Values of Derivatives of L-functions

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L-functions of modular forms

Let S_k be the space of cuspforms of weight k ($k \geq 4$) for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Each $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k$ has an associated L-series:

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \mathrm{Re}(s) \gg 0,$$

which has an analytic continuation to $\mathbb{C}$ and satisfies

$$(2\pi)^{-s}\Gamma(s)L(f,s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)L(f,k-s).$$

The *completed L-function* is denoted by

$$\Lambda(f,s) := (2\pi)^{-s}\Gamma(s)L(f,s).$$

Periods of modular forms

For each n with $0 \le n \le k-2$ and $f \in S_k$, the n^{th} *period* of f is given by

$$r_n(f) := \int_0^{i\infty} f(z) z^n dz = i^{n+1} \Lambda(f, n+1).$$

Each r_n can be regarded as an element in $S_k^* := \mathrm{Hom}_{\mathbb{C}}(S_k, \mathbb{C})$.

There are some natural questions to be asked:

1. Do r_n's span the whole space S_k^*?
2. What are the linear relations among r_n's?
3. The linear independence of a subset of $\{r_n\}_{0 \leq n \leq k-2}$?

Theorem (Eichler-Shimura-Manin) ([8] and [5])

Let $k \geq 4$ be even and S_k be the space of cuspforms. Then

1. $\mathrm{Span}\{r_1, \ldots, r_{k-3}\} = S_k^*$,
2. $\mathrm{Span}\{r_0, \ldots, r_{k-2}\} = S_k^*$.

Furthermore, for any $0 \leq t \leq k-2$ we have

1. $r_t(f) + (-1)^t r_{k-2-t}(f) = 0$,
2. $(-1)^t r_t(f) + \sum_{\substack{0 \leq m \leq t \\ m \equiv 0 \mod 2}} \binom{t}{m} r_{k-2-t+m}(f) + \sum_{\substack{0 \leq m \leq k-2-t \\ m \equiv t \mod 2}} \binom{k-2-t}{m} r_m(f) = 0$
3. $\sum_{\substack{1 \leq m \leq t \\ m \equiv 1 \mod 2}} \binom{t}{m} r_{k-2-t+m}(f) + \sum_{\substack{0 \leq m \leq k-2-t \\ m \not\equiv t \mod 2}} \binom{k-2-t}{m} r_m(f) = 0.$

Linear independence of periods

Theorem (Fukuhara, 2007) [3]

Let $4i \pm 1 := \begin{cases} 4i+1 & k \equiv 2 \pmod 4 \\ 4i-1 & k \equiv 0 \pmod 4 \end{cases}$ and $d_k = \dim_{\mathbb{C}} S_k$. Then

$$\{r_{4i\pm 1} : i = 1, \ldots, d_k\}$$

form a basis for S_k^*.

Theorem (Lei, Ni, Xue, 2023) [7]

Let $n \geq 1$. For sufficiently large k, if $2 \leq \ell_1 < \ell_2 < \ldots < \ell_n \leq \frac{k}{2} - 1$ are even integers, then the set of even periods $\{r_{\ell_i}\}_{i=1}^n$ of S_k is linearly independent.

Theorem (Lei, Ni, Xue, 2023) [6]

Let $n \geq 1$. For sufficiently large k, if $2 \leq \ell_1 < \ell_2 < \cdots < \ell_n \leq \frac{k}{2}$ are even integers, then the set of odd periods $\{r_{\ell_i - 1}\}_{i=1}^n$ on S_k is linearly independent.

Motivation: Build an Eichler-Shimura theory for "derivative period"

Definition

For each n with $0 \leq n \leq k-2$ and $f \in S_k$, we define the "derivative period":

$$D_n(f) = i^{n+1}\Lambda'(f, n+1).$$

Then we can ask the same questions.

1. Do D_n's span the whole space S_k^*?
2. What are the linear relations between D_n?
3. The linear independence of a subset of $\{D_n\}_{0 \leq n \leq k-2}$.

In this talk, we will give a partial answer to the second question.

Short answer

Linear relations between D_n's come from the second Eichler-Shimura cohomology groups.

Eichler-Shimura cohomology

Let $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. Denote by V_{k-2} the abelian group of homogeneous polynomials of degree $k-2$ in variables X and Y with coefficients in $\mathbb{C}$. Then V_{k-2} is a $\mathbb{C}[\mathrm{PSL}_2(\mathbb{Z})]$-module:

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} P\right)(X, Y) := P(aX + cY, bX + dY).$$

We study the second Eichler-Shimura cohomology

$$H^2(\mathrm{PSL}_2(\mathbb{Z}), V_{k-2}) := \mathrm{Ext}^2_{\mathbb{C}[\mathrm{PSL}_2(\mathbb{Z})]}(\mathbb{C}, V_{k-2}).$$

How to make derivative appear?

For even integer $k \geq 4$ and $f \in S_k$ we have

$$\int_0^{\infty} f(iy)y^s \log y \, dy = \Lambda'(f, s+1),$$

by differentiating both sides of Mellin transform with respect to s.

Thanks to Goldfeld [4] and Diamantis [2]'s work, we are able to construct a 2-cocycle $\varphi \in Z^2(\mathrm{PSL}_2(\mathbb{Z}), V_{k-2})$, that is, for any g_1, g_2 and $g_3 \in \mathrm{PSL}_2(\mathbb{Z})$,

$$g_1\varphi(g_2, g_3) - \varphi(g_1 g_2, g_3) + \varphi(g_1, g_2 g_3) - \varphi(g_1, g_2) = 0.$$

Trick: the two-cocycle condition is satisfied if it is a two-coboundary.

Construction of 2-cocyle

- For each $f \in S_k$ we define a one-cochain $I_f \in C^1(\mathrm{PSL}_2(\mathbb{Z}), V_{k-2})$

$$I_f(\gamma) := \int_0^{\gamma(0)} f(z)(Xz+Y)^{k-2} u(z)dz$$

for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and u is given by

$$u(z) := \log(\Delta^2(z)), \quad \Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

- Applying the differential map to get a two-coboundary

$$\varphi_f := d^2 I_f \in B^2(\mathrm{PSL}_2(\mathbb{Z}), V_{k-2}) \subseteq Z^2(\mathrm{PSL}_2(\mathbb{Z}, V_{k-2}).$$

More explicitly, we have

$$\varphi_f(g_1, g_2) = g_1 \cdot \int_0^{g_2(0)} f(z)(Xz+Y)^{k-2}(u(z)-u(g_1 z))dz.$$

Proposition (see also [2, Proposition 1])

Let $k \geq 4$ be even and $f \in S_k$. Define

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

1. $\varphi_f(\sigma, \sigma) = -24 \sum_{\ell=0}^{k-2} \binom{k-2}{\ell} (-1)^{k-2-\ell} D_\ell(f) Y^\ell X^{k-2-\ell}$.
2. $\varphi_f(\tau, \tau) = -24 \sum_{\ell=0}^{k-2} \binom{k-2}{\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^{\ell-j} D_{k-2-\ell+j}(f) Y^{k-2-\ell} X^\ell$

Define the matrices

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Recall the well-known [1] free-product description

$$\mathrm{PSL}_2(\mathbb{Z}) = \langle \sigma \rangle * \langle \tau \rangle = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z},$$

which will ultimately lead to the following.

Mayer-Vietoris for $\mathrm{PSL}_2(\mathbb{Z})$ ([11, Proposition 4.2.2])

Let $G = \mathrm{PSL}_2(\mathbb{Z})$, $H_1 = \langle \sigma \rangle$, $H_2 = \langle \tau \rangle$ and $M = V_{k-2}$. Then the sequences

$$0 \to M^G \to M^{H_1} \oplus M^{H_2} \to M \to H^1(G, M) \xrightarrow{\mathrm{Res}} H^1(H_1, M) \oplus H^1(H_2, M) \to 0$$

are exact. For all $i \geq 2$ there are isomorphisms

$$\mathrm{Res} : H^i(G, M) \cong H^i(H_1, M) \oplus H^i(H_2, M).$$

Computing the cohomology group II

Recall that $H^0(G, M) = M^G := \{m \in M : gm = m \text{ all } g \in G\}$. For a finite cyclic group $G = \langle\sigma\rangle$ of order n and a G-module M, let $N_\sigma : M \to M$ be the G-homomorphism defined as

$$N_\sigma(m) = (1 + \sigma + \cdots + \sigma^{n-1}) \cdot m.$$

A finite cyclic group is well-known to have periodic cohomology.

Lemma [10, Theorem 9.27]

Let $G = \langle\sigma\rangle$ be a finite cyclic group of order n and M be a G-module. Then for all $n \geq 1$,

$$H^0(G, M) = M^G,$$
$$H^{2n-1}(G, M) = \ker N_\sigma/(\sigma - 1)M,$$
$$H^{2n}(G, M) = M^G/N_\sigma M.$$

Now, the structure the second Eichler-Shimura cohomology group is clear.

Lemma

Let $G = \mathrm{PSL}_2(\mathbb{Z}), H_1 = \langle\sigma\rangle$, $H_2 = \langle\tau\rangle$ and $M = V_{k-2}$. Then

$$H^2(G, M) \cong M^{H_1}/N_\sigma M \oplus M^{H_2}/N_\tau M.$$

Eichler-Shimura relations intrinsically

Theorem 1 (Ni. 2024)

Let $k \geq 4$ be even and $f \in S_k$. Define

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$\varphi_f(\sigma, \sigma) = \sigma \cdot \varphi_f(\sigma, \sigma), \quad \varphi_f(\tau, \tau) = \tau \cdot \varphi_f(\tau, \tau). \tag{1}$$

Proof.

It is immediate from

$$H^2(G, M) \cong M^{H_1}/N_\sigma M \oplus M^{H_2}/N_\tau M$$

and the fact that $\varphi_f \in Z^2(G, M)$. □

Theorem 2 (Ni. 2024)

Let $k \geq 4$ be even and $f \in S_k$. Then for any $\ell = 0, ..., k-2$ we have

(ES1) $$D_{k-2-\ell}(f) - (-1)^{k-2-\ell} D_\ell(f) = 0,$$

(ES2) $$\sum_{\substack{0 \leq j \leq \ell \\ j \equiv 1 \pmod 2}} \binom{\ell}{j} D_{k-2-\ell+j}(f) - \sum_{\substack{0 \leq j \leq k-2-\ell \\ j \not\equiv \ell \pmod 2}} \binom{k-2-\ell}{j} D_j(f) = 0,$$

(ES3) $$\sum_{\substack{0 \leq j \leq \ell \\ j \equiv 0 \pmod 2}} \binom{\ell}{j} D_{k-2-\ell+j}(f) - \sum_{\substack{0 \leq j \leq k-2-\ell \\ j \equiv \ell \pmod 2}} \binom{k-2-\ell}{j} D_j(f) = 0.$$

Proof.

Expanding the equations

$$\varphi_f(\sigma, \sigma) = \sigma \cdot \varphi_f(\sigma, \sigma) \quad \text{and} \quad \varphi_f(\tau, \tau) = \tau \cdot \varphi_f(\tau, \tau),$$

we get the results by comparing the $X^\ell Y^{k-2-\ell}$-coefficients of both sides. □

Conjecture

Recall that both the odd periods $\{r_{2i+1}\}_{i=0}^{k/2-2}$ and the even periods $\{r_{2i}\}_{i=0}^{k/2-1}$ span the whole space S_k^*. So it is natural ask the following question:

Conjecture

Suppose $k \geq 4$ is an even integer. Then

$$\mathrm{Span}\left(\{D_{2i+1}\}_{i=0}^{k/2-2}\right) = S_k^* \quad \text{and} \quad \mathrm{Span}\left(\{D_{2i}\}_{i=0}^{k/2-1}\right) = S_k^*.$$

Work in progress

Generalize Theorem 1 and 2 to $S_k(\Gamma_0(N))$.

Idea of the proof

1. We relate the derivatives to cohomology in a similar way:

$$I_f(\gamma) : \int_0^{\gamma(0)} f(z)(Xz+Y)^{k-2}u(z)dz, \quad \varphi_f := d^2 I_f.$$

where

$$u(z) = \log(\Delta(z)\cdot\Delta(Nz)).$$

2. We need to study the cohomology group $H^2(\Gamma_0(N), V_{k-2})$.

- Reduction to level one by Shapiro's Lemma:

$$H^n(G, \mathrm{Coind}_H^G(A)) \cong H^n(H, A).$$

- To obtain the linear relations, we need the explicit inverse of the Shapiro's map. Naidu's paper *"Categorical Morita equivalence for group-theoretical categories"* [9] gives us the idea on how to overcome this difficulty.

Thank you!

"And the world is passing away along with its desires, but whoever does the will of God abides forever." – 1 John 2:17

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