Nonvanishing of the Second Coefficient of General Hecke Polynomials

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Presentation Overview

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Abstract

Given a Hecke operator $T_m(N, k, \chi)$ (where N is coprime to m and $k \geq 2$), a well-known open problem is to determine when the trace of this Hecke operator is vanishing. Recall that the trace of $T_m(N, k, \chi)$ comes from the first coefficient of its associated Hecke polynomial. So we consider a slightly different problem: when will the second coefficient of the $T_m(N, k, \chi)$ Hecke polynomial vanish? In this paper, we show that for any given m, the second coefficient of the $T_m(N, k, \chi)$ Hecke polynomial is nonvanishing for all but finitely many (N, k, χ). We also compute the complete list of such (N, k) for T_3 with trivial character.

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- Such an f has character χ if $f(\gamma z)(cz + d)^{-k} = \chi(d) f(z)$ $\forall z \in \mathbb{H}, \ \ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.

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- Such an f is a cuspidal if $f(z) \to 0$ as $\mathrm{Im}(z) \to \infty$
- We let $S_k(\Gamma_0(N), \chi)$ denote the space of all such cuspidal modular forms of weight k and level N with character χ.

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- Recall the family of Hecke operators on this space:

$$T_m(N,k,\chi)\colon S_k(\Gamma_0(N),\chi) \longrightarrow S_k(\Gamma_0(N),\chi)$$
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- $$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n \in S_k(\Gamma_0(1))$$

  is the unique eigenvector for each $T_m(1,12)$ (with eigenvalue $\tau(m)$). Thus Tr $T_m(1,12) = \tau(m)$

Coefficients of Hecke Polynomials

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- We instead consider a slightly different problem: the nonvanishing of $a_2(m,N,k)$. And in fact, we generalize slightly and consider $a_2(m,N,k,\chi)$.
- Last year, Clayton et. al. [1] dealt with the nonvanishing of $a_2(m,N,k)$ in the case of $N = 1$ and in the case of $m = 2$ (both with trivial character).

Coefficients of Hecke Polynomials

In this presentation, we will show the following theorem

Theorem 1

Let $m \geq 1$ be fixed and consider N coprime to m, $k \geq 2$, and χ a Dirichlet character such that $\chi(-1) = (-1)^k$. Then $a_2(m, N, k, \chi)$ vanishes for only finitely many (N, k, χ).

We will also compute the complete list of (N, k) (with trivial character) for which $a_2(3, N, k)$ vanishes.

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A formula for $a_2(m, N, k, \chi)$

First, we compute a formula for $a_2(m, N, k, \chi)$ in terms of traces.

Formula 2

For convenience of notation, let T_m *denote* $T_m(N, k, \chi)$. *Then*

$$a_2(m, N, k, \chi) = \frac{1}{2}\left[(\operatorname{Tr} T_m)^2 - \sum_{d|m} \chi(d) d^{k-1} \operatorname{Tr} T_{m^2/d^2}\right]$$

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Proof.

Let $\lambda_1 \ldots \lambda_n$ be the eigenvalues of T_m. Then by definition of the characteristic polynomial, we have

$$a_2(m, N, k, \chi) = \sum_{1 \le i < j \le n} \lambda_i \lambda_j$$

Proof.

But note

$$\sum_{1\leq i<j\leq n} \lambda_i\lambda_j = \frac{1}{2}\left[\left(\sum_{1\leq i\leq n}\lambda_i\right)^2 - \sum_{1\leq i\leq n}\lambda_i^2\right]$$
$$= \frac{1}{2}\left[(\mathrm{Tr}\ T_m)^2 - \mathrm{Tr}\ T_m^2\right]$$

And recall how the Hecke operators compose:

$$T_m^2 = \sum_{d|m}\chi(d)d^{k-1}T_{m^2/d^2}$$

Thus

$$a_2(m,N,k,\chi) = \frac{1}{2}\left[(\mathrm{Tr}\ T_m)^2 - \mathrm{Tr}\ T_m^2\right]$$
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The Eichler-Selberg trace formula

Let $m \geq 1$, $N \geq 1$, $k \geq 2$, and χ a Dirichlet character modulo N such that $\chi(-1) = (-1)^k$. Then

$$\text{Tr } T_m(N, k, \chi) = A_{1,m} - A_{2,m} - A_{3,m} + A_{4,m} \tag{4.1}$$

where

$$A_{1,m} = \chi(\sqrt{m}) \frac{k-1}{12} \psi(N) m^{k/2-1} \tag{4.2}$$

$$A_{2,m} = \frac{1}{2} \sum_{t^2 < 4m} U_{k-1}(t, m) \sum_n h_w \left(\frac{t^2 - 4m}{n^2} \right) \mu(t, n, m) \tag{4.3}$$

$$A_{3,m} = \frac{1}{2} \sum_{d|m} \min(d, m/d)^{k-1} \sum_\tau \phi(\gcd(\tau, N/\tau)) \chi(y) \tag{4.4}$$

$$A_{4,m} = \begin{cases} \sum\limits_{\substack{c|m \\ (N, m/c)=1}} c & \text{if } k = 2 \text{ and } \chi = \chi_0 \\ 0 & \text{if } k > 2 \text{ or } \chi \neq \chi_0 \end{cases}. \tag{4.5}$$

- $\chi(\sqrt{m})$ is interpreted as 0 if $\sqrt{m} \notin \mathbb{Z}$.
- $\psi(N) = [\,\mathrm{SL}\,2(\mathbb{Z}) : \Gamma_0(N)\,] = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$
- The outer summation in $A_{2,m}$ runs over all $t \in \mathbb{Z}$ such that $t^2 < 4m$. Note that the terms corresponding to $t = t_0$ and $t = -t_0$ coincide.
- $U_{k-1}(t, m)$ denotes the Lucas sequence of the first kind. In particular, $U_{k-1}(t, m) = \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}}$ where $\rho, \bar{\rho}$ are the two roots of the polynomial $X^2 - tX + m$.
- The inner summation in $A_{2,m}$ runs through all positive integers n such that $n^2 \,|\, (t^2 - 4m)$, and $\frac{t^2-4m}{n^2} \equiv 0, 1 \pmod 4$.
- $h_w\left(\frac{t^2-4m}{n^2}\right)$ is the weighted class number of the imaginary quadratic order with discriminant $\frac{t^2-4m}{n^2}$. This is the usual class number, divided by 2 (respectively 3) if the discriminant is -4 (respectively -3).

- $\mu(t,n,m) = \dfrac{\psi(N)}{\psi(N/N_n)} \sum'_{c \bmod N} \chi(c)$, where $N_n = \gcd(N,n)$, and the summation runs through all elements c of $(\mathbb{Z}/N\mathbb{Z})^\times$ which lift to solutions of $c^2 - tc + m \equiv 0 \pmod{NN_n}$.
- The outer summation for $A_{3,m}$ runs over all positive divisors d of m. Note that the terms corresponding to $d = d_0$ and $d = m/d_0$ coincide.
- The inner summation for $A_{3,m}$ runs over all positive divisors τ of N such that $\gcd(\tau, N/\tau)$ divides $\gcd(N/N_\chi, d - m/d)$. Here N_χ is the conductor of χ.
- ϕ is the Euler totient function.
- y is the unique integer modulo $\operatorname{lcm}(\tau, N/\tau)$ determined by the congruences $y \equiv d \mod \tau$ and $y \equiv \frac{m}{d} \mod \frac{N}{\tau}$.
- χ_0 denotes the trivial character modulo N.
- Throughout, remember that χ is a character modulo N, so $\chi(a) = 0$ if $\gcd(a,N) > 1$ (even in the trivial character case).

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Lemma 3

Recall that $\psi(N) = N\prod_{p|N}\left(1+\frac{1}{p}\right)$, *and let* $\omega(N)$ *denote the number of distinct prime divisors of* N. *Then let*

$$\theta_1(N) := \frac{2^{\omega(N)}\sqrt{N}}{\psi(N)}$$

$$\theta_2(N) := \frac{(2^{\omega(N)})^2}{\psi(N)}$$

$$\theta_3(N) := \frac{2^{\omega(N)}}{\psi(N)}$$

Then each $\theta_i(N) \to 0$ *as* $N \to \infty$. *And in particular, we have*

$N \geq$	1	43	571	8,800	150,000
$\theta_1(N) \leq$	1.00	0.465	0.257	0.133	0.0607
$\theta_2(N) \leq$	1.34	0.445	0.149	0.0424	0.00941
$\theta_3(N) \leq$	1.00	0.0556	0.00926	0.00133	0.000147

Proof.

Note that every prime other than $2, 3, 5, 7$ is ≥ 8. Thus $\omega(N) \leq 4 + \log_8(N)$ and so $2^{\omega(N)} \leq 2^{4+\log_8(N)} \leq 16 \cdot N^{1/3} = O(N^{1/3})$. Thus since $\psi(N) \geq N$, we have that

$$\theta_1(N) = \frac{2^{\omega(N)}\sqrt{N}}{\psi(N)} = O(\frac{N^{5/6}}{N}) \longrightarrow 0$$

$$\theta_2(N) = \frac{(2^{\omega(N)})^2}{\psi(N)} = O(\frac{N^{2/3}}{N}) \longrightarrow 0$$

$$\theta_3(N) = \frac{2^{\omega(N)}}{\psi(N)} = O(\frac{N^{1/3}}{N}) \longrightarrow 0$$

□

Lemma 4

Suppose that $4m - t^2 > 0$. *Then*

$$|U_{k-1}(t, m) \cdot \mu(t, n, m)| \leq 2\psi(n)2^{\omega(N)} m^{(k-1)/2}$$

Lemma 5

Let $\Sigma(N, m, d)$ *denote the inner summation for* $A_{3,m}$*, i.e.*

$$\Sigma(N, m, d) = \sum_{\tau}{}' \phi(\gcd(\tau, N/\tau))\chi(y)$$

Then

$$|\Sigma(N, m, d)| \leq \begin{cases} \left|d - \frac{m}{d}\right| \cdot 2^{\omega(N)} & \text{if } d \neq \sqrt{m}, \\ \sqrt{N} \cdot 2^{\omega(N)} & \text{in general} \end{cases}$$

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Size of Tr T_m

In this section, we will show the main theorem. We split into two cases: when $\sqrt{m} \notin \mathbb{Z}$ and when $\sqrt{m} \in \mathbb{Z}$.

Lemma 6

Let m be fixed where $\sqrt{m} \notin \mathbb{Z}$. Then for all N coprime to m, $k \geq 2$, and χ a Dirichlet character modulo N with $\chi(-1) = (-1)^k$, we have

$$\text{Tr } T_m(N, k, \chi) = O(2^{\omega(N)} m^{k/2})$$

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Lemma 7

Let m be fixed where $\sqrt{m} \in \mathbb{Z}$. Then for all N coprime to m, $k \geq 2$, and χ a Dirichlet character modulo N with $\chi(-1) = (-1)^k$, we have

$$\text{Tr } T_m(N, k, \chi) = \chi(\sqrt{m}) \frac{k-1}{12} \psi(N) m^{k/2-1} + O(\sqrt{N} 2^{\omega(N)} m^{k/2})$$

The main theorem when $\sqrt{m} \notin \mathbb{Z}$

Theorem 8

Let m be fixed where $\sqrt{m} \notin \mathbb{Z}$, and consider N coprime to m, $k \geq 2$, and χ a Dirichlet character modulo N with $\chi(-1) = (-1)^k$. Then $a_2(m, N, k, \chi)$ is nonvanishing (and in particular negative if $\chi = \chi_0$) for all but finitely many (N, k, χ).

The main theorem when $\sqrt{m} \notin \mathbb{Z}$

Proof.

We have

$$a_2(m, N, k) = \frac{1}{2}\left[(\operatorname{Tr} T_m)^2 - \sum_{d|m} d^{k-1} \operatorname{Tr} T_{m^2/d^2}\right]$$

Now, note every term inside the sum has m^2/d^2 a perfect square. Thus for each term in the sum, we have by Lemma 7,

$$\begin{aligned} d^{k-1} \operatorname{Tr} T_{m^2/d^2} &= \dots \\ &= d\,\frac{k-1}{12}\psi(N)m^{k-2} + O(\sqrt{N}2^{\omega(N)}m^k) \end{aligned}$$

The main theorem when $\sqrt{m} \notin \mathbb{Z}$

Proof.

So we have

$$
\begin{aligned}
& a_2(m, N, k) \\
&= \frac{1}{2}\left[(\operatorname{Tr} T_m)^2 - \sum_{d \mid m} d^{k-1} \operatorname{Tr} T_{m^2/d^2}\right] \\
&= \frac{1}{2}\left[(\operatorname{Tr} T_m)^2 - \sum_{d \mid m}\left[d\,\frac{k-1}{12}\psi(N) m^{k-2} + O(\sqrt{N} 2^{\omega(N)} m^k)\right]\right] \\
&= \frac{1}{2}\left[(\operatorname{Tr} T_m)^2 - \frac{k-1}{12}\psi(N) m^{k-2}\sigma_1(m) + O(\sqrt{N} 2^{\omega(N)} m^k)\right]
\end{aligned}
$$

The main theorem when $\sqrt{m} \notin \mathbb{Z}$

Proof.

Then since $\sqrt{m} \notin \mathbb{Z}$, we apply Lemma 6 to Tr T_m to obtain

$$
\begin{aligned}
& a_2(m, N, k) \\
&= \frac{1}{2}\left[O(2^{\omega(N)} m^{k/2})^2 - \frac{k-1}{12}\psi(N) m^{k-2}\sigma_1(m) - O(\sqrt{N} 2^{\omega(N)} m^k)\right] \\
&= \frac{1}{2}\left[-\frac{k-1}{12}\psi(N) m^{k-2}\sigma_1(m) + O(\sqrt{N} 2^{\omega(N)} m^k)\right] \\
&= \frac{\psi(N) m^{k-2}\sigma_1(m)}{2}\left[-\frac{k-1}{12} + O(\theta_1(N))\right]
\end{aligned}
$$

This means that $a_2(m, N, k)$ will be negative for all but finitely many (N, k). □

The main theorem when $\sqrt{m} \in \mathbb{Z}$

Theorem 9

Let m be fixed where $\sqrt{m} \in \mathbb{Z}$, and consider $N \geq 1$ coprime to m, $k \geq 2$, and χ a Dirichlet character modulo N with $\chi(-1) = (-1)^k$. Then $a_2(m, N, k, \chi)$ is nonvanishing (and in particular positive if $\chi = \chi_0$) for all but finitely many (N, k, χ).

Proof.

Just like last time, we have

$$
\begin{aligned}
&a_2(m, N, k) \\
&= \frac{1}{2}\left[(\mathrm{Tr}\ T_m)^2 - \frac{k-1}{12}\psi(N)m^{k-2}\sigma_1(m) + O(\sqrt{N}2^{\omega(N)}m^k)\right] \\
&= \frac{1}{2}\left[(\mathrm{Tr}\ T_m)^2 + O(k\psi(N)m^k)\right]
\end{aligned}
$$

The main theorem when $\sqrt{m} \in \mathbb{Z}$

Proof.

Then since $\sqrt{m} \in \mathbb{Z}$, we apply Lemma 7 to Tr T_m, and we have

$$
\begin{aligned}
(\mathrm{Tr}\, T_m)^2 &= \left(\frac{k-1}{12} \psi(N) m^{k/2-1} + O(\sqrt{N} 2^{\omega(N)} m^{k/2}) \right)^2 \\
&= \dots \\
&= \frac{(k-1)^2}{144} \psi(N)^2 m^{k-2} + O(k \psi(N) \sqrt{N} 2^{\omega(N)} m^k)
\end{aligned}
$$

The main theorem when $\sqrt{m} \in \mathbb{Z}$

Proof.

which yields

$$\begin{aligned} a_2(m,N,k) &= \frac{1}{2}\left[(\mathrm{Tr}\, T_m)^2 + O(k\psi(N)m^k)\right] \\ &= \frac{1}{2}\left[\frac{(k-1)^2}{144}\psi(N)^2 m^{k-2} + O(k\psi(N)\sqrt{N}2^{\omega(N)}m^k)\right] \\ &= \frac{k\psi(N)^2 m^{k-2}}{2}\left[\frac{(k-1)^2}{144k} + O(\theta_1(N))\right] \end{aligned}$$

This means that $a_2(m,N,k)$ will be positive for all but finitely many (N,k). □

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Nonvanishing of $a_2(3, N, k)$

The bounds given here are very computable. In particular we go through the details for $m = 3$ to compute the specific constants hiding in the $O(\theta_1(N))$ term from theorem 8. This allows us to compute all the (N, k) (using trivial character) for which $a_2(3, N, k)$ vanishes.

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Note by Formula 2, we have

$$a_2(3, N, k) = \frac{1}{2}\left[(\mathrm{Tr}\ T_3)^2 - \mathrm{Tr}\ T_9 - 3^{k-1}\,\mathrm{Tr}\ T_1\right]$$

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So we give three Lemmas to get bounds on the three terms in the above formula.

Bounds on error terms

Lemma 10

$$\frac{(\mathrm{Tr}\, T_3)^2}{\psi(N)3^k} \leq \frac{448 + 160\sqrt{3}}{27}\theta_2(N)$$

Bounds on error terms

Lemma 11

$$\left| \frac{\text{Tr } T_9 - A_{1,9}}{\psi(N)3^k} \right| \leq \frac{183}{18}\theta_3(N) + \frac{1}{6}\theta_1(N) \qquad (7.1)$$

where

$$A_{1,9} = \frac{k-1}{108}\psi(N)3^k$$

Bounds on error terms

Lemma 12

$$\left|\frac{\text{Tr } T_1 - A_{1,1}}{\psi(N)}\right| \leq \frac{5}{3}\theta_3(N) + \frac{1}{2}\theta_1(N)$$

where

$$A_{1,1} = \frac{k-1}{12}\psi(N)$$

Nonvanishing of $a_2(3, N, k)$

Theorem 13

Consider N coprime to 3 and $k \geq 2$ even. Then $a_2(3, N, k)$ vanishes only for the values of (N, k) given in Table 14.

Nonvanishing of $a_2(3, N, k)$

Theorem 13

Consider N coprime to 3 and $k \geq 2$ even. Then $a_2(3, N, k)$ vanishes only for the values of (N, k) given in Table 14.

Proof.

By Formula 2, we have

$$
\begin{aligned}
a_2(3, N, k) &= \frac{1}{2}\left[(\mathrm{Tr}\ T_3)^2 - \mathrm{Tr}\ T_9 - 3^{k-1}\,\mathrm{Tr}\ T_1\right] && (7.2)\\
&= \dots && (7.3)\\
&= \frac{\psi(N)3^k}{2}\left[-\frac{k-1}{27} + E(N, k)\right] && (7.4)
\end{aligned}
$$

Nonvanishing of $a_2(3, N, k)$

Proof.

where $E(N, k)$ denotes the three error terms

$$|E(N,k)| = \left|\frac{(\mathrm{Tr}\ T_3)^2}{\psi(N)3^k} - \left(\frac{\mathrm{Tr}\ T_9 - A_{1,9}}{\psi(N)3^k}\right) - \frac{1}{3}\left(\frac{\mathrm{Tr}\ T_1 - A_{1,1}}{\psi(N)}\right)\right| \quad (7.5)$$

$$\leq \dots \quad (7.6)$$

$$= \frac{448 + 160\sqrt{3}}{27}\theta_2(N) + \frac{193}{18}\theta_3(N) + \frac{1}{3}\theta_1(N) \quad (7.7)$$

Then using the numerical θ_i bounds given in Lemma 3, this shows that $a_2(3, N, k) < 0$ for $N \geq 63{,}000{,}000$, $k \geq 2$; $N \geq 2{,}700{,}000, k \geq 4$; $N \geq 150{,}000, k \geq 10$; $N \geq 8{,}800$, $k \geq 34$; $N \geq 571, k \geq 116$; $N \geq 43, k \geq 344$; and $N \geq 1, k \geq 1272$.

We then check the finite number of cases left by computer, which yields the complete list given in Table 14. □

Table of (N, k) with vanishing a_2 coefficient

Table 14

The values of (N, k) for which $a_2(3, N, k)$ vanishes				
$(1, 2)$	$(1, 4)$	$(1, 6)$	$(1, 8)$	$(1, 10)$
$(1, 12)$	$(1, 14)$	$(1, 16)$	$(1, 18)$	$(1, 20)$
$(1, 22)$	$(1, 26)$	$(2, 2)$	$(2, 4)$	$(2, 6)$
$(2, 8)$	$(2, 10)$	$(4, 2)$	$(4, 4)$	$(4, 6)$
$(5, 2)$	$(5, 4)$	$(5, 6)$	$(7, 2)$	$(7, 4)$
$(8, 2)$	$(8, 4)$	$(10, 2)$	$(11, 2)$	$(13, 2)$
$(14, 2)$	$(16, 2)$	$(17, 2)$	$(19, 2)$	$(20, 2)$
$(25, 2)$	$(32, 2)$	$(34, 2)$	$(44, 2)$	$(49, 2)$
$(56, 2)$	$(64, 2)$	$(80, 2)$	$(140, 2)$	$(280, 2)$

A conjecture

Conjecture 15 ([1, Conjecture 5.1])

Write the characteristic polynomial for T_m *over* $S_k(\mathrm{SL}_2(\mathbb{Z}))$ *as* $x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots + (-1)^n a_n$. *Then* $a_i \neq 0$ *for* $1 \leq i \leq n$.

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