Evaluation of Convolution Sums and Modular Forms

Brundaban Sahu

National Institute of Science Education and Research, Bhubaneswar

36th Automorphic forms Workshop, OSU, Stillwater, May 20–24, 2024

Outline

Outline

- Convolution sums
- Elementary evaluations
- Modular forms
- Applications

Convolution sums

Convolution sums

- Let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of positive integers.
- For $k, n \in \mathbb{N}$, let $\sigma_k(n) = \sum_{d|n, d\in\mathbb{N}} d^k$
- $\sigma_0(n)$ is the number of divisors of n, $\sigma_1(n)$ is the sum of divisors of n (denote by $\sigma(n)$).

Convolution sums

- Let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of positive integers.
- For $k, n \in \mathbb{N}$, let $\sigma_k(n) = \sum_{d|n, d\in\mathbb{N}} d^k$
- $\sigma_0(n)$ is the number of divisors of n, $\sigma_1(n)$ is the sum of divisors of n (denote by $\sigma(n)$).
- The 1916 paper (On certain arithmetical functions, Trans. Cambridge Phil. Soc. Vol. 22, 159-184) of S. Ramanujan starts with the following:
  "... let
  $$\sum_{r,s}(n) = \sigma_r(0)\sigma_s(n) + \sigma_r(1)\sigma_s(n-1) + \cdots + \sigma_r(n)\sigma_s(0).$$
  In this paper I prove that
  $$\sum_{r,s}(n) = \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\sigma_{r+s+1}(n) + \frac{\zeta(1-r)+\zeta(1-s)}{r+s}n\sigma_{r+s-1}(n) + O(n^{2(r+s+1)/3})$$
  whenever r and s are odd positive integers."
  Here, $\sigma_s(0) = \frac{1}{2}\zeta(-s)$ (Riemann zeta function)

Convolution sums

Convolution sums

- He proved that there is no error term if $(r, s) \in \{(1,1), (1,3), (1,5), (1,7), (1,11), (3,3), (3,5), (3,9), (5,7)\}$
- For above values of (r, s), he explicitly computed the convolution sums.

$$\sum_{1,1}(n) = \frac{5\sigma_3(n) - 6n\sigma(n)}{12}$$

$$\sum_{1,3}(n) = \frac{7\sigma_5(n) - 10n\sigma_3(n)}{80}$$

- The formula for $\sum_{1,1}(n)$ has originally appeared in a letter from Besge to Liouville (1862)
- Appeared also in the work of Glaisher (1884)
- Evaluation of convolution sums by Lahiri (1946)

Convolution sums

Convolution sums

- For $a, b, r, s, n \in \mathbb{N}$, consider the following sum:

$$W_{a,b}^{r,s}(n) := \sum_{\substack{l,m \in \mathbb{N} \\ al+bm=n}} \sigma_r(l)\sigma_s(m).$$

- For $r = s = 1$ we denote it by $W_{a,b}(n) = \sum_{\substack{l,m \in \mathbb{N} \\ al+bm=n}} \sigma(l)\sigma(m)$, $W_{1,a} = W_a$ and $W_{b,1} = W_b$.
- $$W_{1,1}^{1,1}(n) = \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma(n), W_{1,1}^{1,3}(n) = \frac{7}{80}\sigma_5(n) + \frac{(1-3n)}{24}\sigma_3(n) - \frac{1}{240}\sigma(n)$$
- There are various methods to evaluate these convolution sums explicitly either by using elementary evaluation or by using the theory modular forms and quasimodular forms or by (p, k) parametrization etc.

Convolution sums

Convolution sums

- In 1993, Skoruppa studied Eisenstein Series identities using elementary techniques by using a key combinatorial identity.

- Theorem: Let $f : \mathbb{Z}^2 \to \mathbb{C}$ be a function satisfying

$$h(y, y-x) = h(x, y) \quad \forall x, y \in \mathbb{Z}.$$

Then, for any positive integer n, one has

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (h(a,b) - h(a,-b)) = \sum_{\substack{d\in\mathbb{N} \\ d\mid n}} \left(\frac{n}{d} h(d,0) - \sum_{i=0}^{n-1} h(d,i) \right)$$

- Note: LHS - sum over all positive integers a, b, x, y such that $ax + by = n$ and RHS - sum over all positive divisors d of n.

Convolution sums

Convolution sums

- To get $W_1(n)$, Skoruppa considered $h(x,y) = -\frac{1}{2}(x^2 - xy + y^2)$ and then show that the LHS is equal to

$$\sum_{ax+by=n} ab = W_1(n).$$

- In this case, the RHS is equal to

$$\sum_{d|n} \left(-\frac{n}{2}d + \frac{5}{12}d^3 + \frac{1}{12}d\right)$$

Combining these two, we get

$$W_1(n) = \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma(n).$$

Convolution sums

- To get $W_1(n)$, Skoruppa considered $h(x, y) = -\frac{1}{2}(x^2 - xy + y^2)$ and then show that the LHS is equal to

$$\sum_{ax+by=n} ab = W_1(n).$$

- In this case, the RHS is equal to

$$\sum_{d|n}(-\frac{n}{2}d + \frac{5}{12}d^3 + \frac{1}{12}d)$$

Combining these two, we get

$$W_1(n) = \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma(n).$$

- Huard, Ou, Spearman and Williams (2000): Elementary evaluation of convolution sums, K. S. Williams (2013): Number Theory in the spirit of Liouville
- The purpose of this talk to look at a basic result in theory of modular forms and present a simple extension, and as consequence, this gives a unified method to find these types (and more general types) of convolution sums.

Modular forms

Modular forms

- $M_k(N)$= Space of modular forms of weight k for $\Gamma_0(N)$
  $S_k(N)$= Space of cusp forms of weight k for $\Gamma_0(N)$
- Example: For even $k \geq 4$, the Eisenstein series

$$E_k(z) = \frac{1}{\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$$

  is a modular form of weight k for $\Gamma_0(1) = SL_2(\mathbb{Z})$ with Fourier expansion
  $E_k(z) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n$.
- $E_4(z) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \quad E_6(z) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n,$
  $E_8(z) = 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n, \quad E_{10}(z) = 1 - 264 \sum_{n\geq 1} \sigma_9(n) q^n.$
- The well known modular identities $E_8 = E_4^2, E_{10} = E_4 E_6$ give the convolution sums
  $W_{1,1}^{3,3}(n) = \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n)$ and $W_{1,1}^{3,5}(n) = \frac{11}{5040}\sigma_9(n) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(n)$.

Quasimodular forms

Quasimodular forms

- E. Royer (2007) demonstrated the use of quasimodular forms to prove a wide range of these convolution sums.

- The Eisenstein series $E_2(z)$

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}$$

is a quasimodular form of weight 2 and depth 1 for the full modular group. It is a holomorphic function and does not satisfy the invariance property w.r. to the $SL_2(\mathbb{Z})$ action. In fact, we have

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) + \frac{12c}{2\pi i}(cz+d).$$

Quasimodular forms

- E. Royer (2007) demonstrated the use of quasimodular forms to prove a wide range of these convolution sums.

- The Eisenstein series $E_2(z)$

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}$$

is a quasimodular form of weight 2 and depth 1 for the full modular group. It is a holomorphic function and does not satisfy the invariance property w.r. to the $SL_2(\mathbb{Z})$ action. In fact, we have

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) + \frac{12c}{2\pi i}(cz+d).$$

- The number of error (additional) terms in the transformation property is called the depth of the function. For E_2, the depth is 1.

Quasimodular forms

Quasimodular forms

- A complex valued holomorphic function f defined on the upper half-plane $\mathbb{H}$ is called a quasimodular form of weight k, depth s (s is a non-negative integer), if there exist holomorphic functions $f_0, f_1, \ldots f_s$ on $\mathbb{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{i=0}^{s} f_i(z) \left(\frac{c}{cz+d}\right)^i,$$

  for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and such that f_s is holomorphic at the cusps and not identically vanishing.

Quasimodular forms

- A complex valued holomorphic function f defined on the upper half-plane $\mathbb{H}$ is called a quasimodular form of weight k, depth s (s is a non-negative integer), if there exist holomorphic functions $f_0, f_1, \dots f_s$ on $\mathbb{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{i=0}^{s} f_i(z) \left(\frac{c}{cz+d}\right)^i,$$

  for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and such that f_s is holomorphic at the cusps and not identically vanishing.

- The quasimodular forms of depth 0 are exactly modular forms. It is a fact that the depth of a quasimodular form of weight k is less than or equal to $k/2$.

Quasimodular forms

Quasimodular forms

- $M_k^{\leq k/2}(\Gamma_0(N))$= space of quasimodular forms of weight k and of depth less than equal to $k/2$ for the congruence subgroup $\Gamma_0(N)$.

- Theorem (Kaneko and Zagier, 1994)

$$M_k^{\leq k/2}(\Gamma_0(N)) = \oplus_{i=0}^{k/2-1} D^i M_{k-2i}(\Gamma_0(N)) \oplus \mathbb{C} D^{k/2-1} E_2$$

Here D is the differential operator $\frac{1}{2\pi i}\frac{d}{dz}$.

- One can express each quasimodular form of weight k and depth $\leq k/2$ as a linear combination of j-th derivatives of modular forms of weight $k-2j$ on $\Gamma_0(N)$, $0 \leq j \leq k/2-1$ and the $(k/2-1)$-th derivate of the quasimodular form E_2.

Quasimodular forms

Quasimodular forms

- Using structure theorem, one can evaluate sums of the forms

$$W_{a,b}(n) = \sum_{al+bm=n} \sigma(l)\sigma(m)$$

- To evaluate the convolutions sums $W_1(n)$ we consider $E_2(z)E_2(z)$, which is quasimodular form of weight 4 and depth 2.
- Using the structure theorem, one express $E_2(z)E_2(z)$ in terms of basis and finally compute $W_1(n)$.

$$E_2(z)^2 = (1 - 24\sum_{n=1}^{\infty} \sigma(n)q^n)^2 = E_4(z) + 12DE_2(z).$$

- Comparing n-th coefficients both sides, we get

$$-48\sigma(n) + 24^2 W_1(n) = 240\sigma_3(n) - 288n\sigma(n)$$

from which $W_1(n)$ is obtained.

Quasimodular forms

Quasimodular forms

- To evaluate $W_5(n)$, consider the quasimodular form $E_2(z)E_2(5z)$.

- Using the structure theorem, $E_2(z)E_2(5z)$ can be expressed as

$$E_2(z)E_2(5z) = \frac{1}{26}E_4(z) + \frac{25}{26}E_4(5z) - \frac{288}{65}\eta^4(z)\eta^4(5z) + \frac{24}{5}D\Phi_{1,5}(z) + \frac{12}{5}DE_2(z),$$

where $\Phi_{1,5}(z) = \frac{1}{4}(5E_2(5z) - E_2(z))$.

- Comparing n-th Fourier coefficients both sides, we get

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3(n/5) + \left(\frac{1}{24} - \frac{n}{20}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/5) - \frac{1}{130}c_5(n).$$

Quasimodular forms

Quasimodular forms

- Ramakrishnan-S (2013) evaluated $W_{15}(n)$ and $W_{3,5}(n)$ using quasimodular forms and used to compute the representation numbers for the following quadratic forms:

$$(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2) + 5(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2),$$

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2) + 3(x_5^2 + x_6^2 + x_7^2 + x_8^2),$$

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2) + 6(x_5^2 + x_6^2 + x_7^2 + x_8^2).$$

- We evaluated $W_{21}(n)$ and $W_{3,7}(n)$ (BB3- Proceedings) and obtained representation number formulas for the following quadratic forms:

$$(x_1^2 + x_1x_2 + 2x_2^2 + x_3^2 + x_3x_4 + 2x_4^2) + j(x_5^2 + x_5x_6 + 2x_6^2 + x_7^2 + x_7x_8 + 2x_8^2),$$

for $j = 1, 2, 3, 4$.

Table: List of $W_N(n)$

N	Author(s)	Year
1	Besge, Glaisher, Ramanujan,	1862/1885/1916/1993
1, –	Skoruppa	1993
2, 3, 4	Huard, Ou, Spearman & Williams	2000
5, 7	Lemire & Williams	2006
6	Alaca & Williams	2007
8, 9	Williams	2005/2006
12, 16, 18, 24	Alaca, Alaca & Williams	2007
10, 11, 13, 14	Royer	2007
23	Chan & Cooper	2008
15	Ramakrishnan & Sahu	2013
20	Cooper & Ye	2014
25	Xia, Tian & Yao	2015
27, 32	Alaca & Kesicioglu	2015
36	Ye	2015
21	Ramakrishnan & Sahu	2016
..	..	..
59, 71	Cho	2020

Table: List of $W_{a,b}^{1,3}(n)$

(a, b)	Author(s)	Year
(1, 1)	Ramanujan/Lahiri/	1916/ 1947
	Cheng &Wiiliams/ Royer	2005/2007
(1, 2), (2, 1)	Melfi, Huard et. al,	1989, 2000
	Cheng &Wiiliams/ Royer	2005/ 2007
(1, 3) (3, 1)	Yao & Xia	2013
(1, 4) (4, 1)	Cheng & Williams	2005
(1, 6), (6, 1), (2, 3), (3, 2)	Kokluce	2022

Ramanujan-Serre derivative

Ramanujan-Serre derivative

- The derivative of a modular form is not a modular form.

  If we differentiate the transformation formula for f

  $$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

  w.r.t z, we get

  $$f'\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2} f'(z) + kc(cz+d)^{k+1} f(z).$$

  Nevertheless, there are many interesting connections between differential operators and the theory of modular forms.
- Use E_2 and differential operator D to create new modular forms.

- f is a modular form of weight k (of any level N) then

  $$\delta_k(f) := Df - \frac{k}{12} E_2(z) f(z)$$

  is a modular form of weight $k+2$.

Simple extension of Ramanujan-Serre derivative map

Simple extension of Ramanujan-Serre derivative map

- For positive integers a, b, define the differential operator $\delta_{k,a,b}$

$$\delta_{k,a,b}(f) := bDf(bz) - \frac{k}{12}aE_2(az)f(bz)$$

- Theorem: Let a, b, k, M be positive integers and f be a modular form in $M_k(M, \chi)$, where χ is a Dirichlet character modulo M such that $\chi(-1) = (-1)^k$. Then $\delta_{k,a,b}(f)$ is a modular form in $M_{k+2}(N, \chi)$, where $N = l.c.m(Mb, a)$.

  Further, for a, b with $g.c.d(a, b) = 1$ the function

$$\frac{6a}{b}DE_2(az) + \frac{6b}{a}DE_2(bz) - E_2(az)E_2(bz)$$

  is a modular form in the space $M_4(ab)$.

- For $a = b = 1$, the differential operator $\delta_{k,1,1} = \delta_k$.
- Ramanujan: $E_2^2(z) - 12DE_2(z)$ is a modular form of weight 4 (equals to $E_4(z)$). The second part extends to congruence subgroups.

Application 1: Convolution sums

Application 1: Convolution sums

- In particular, considering $f(z) = E_k(z) \in M_k$ with $gcd(a,b) = 1$

$$bDE_k(bz) - \frac{k}{12} aE_2(az)E_k(bz) \in M_{k+2}(ab).$$

- Let's denote a basis of $M_{k+2}(ab)$ by $f_i(z), 1 \leq i \leq \lambda_{k+2}(ab)$ where $\lambda_{k+2}(ab)$ is the dimension of $M_{k+2}(ab)$ whose n-th Fourier coefficients are given by $a_i(n)$.
- Hence, there are constants $\alpha_i, 1 \leq i \leq \lambda_{k+2}(ab)$ such that

$$bDE_k(bz) - \frac{k}{12} aE_2(az)E_k(bz) = \sum_{i=1}^{\lambda_{k+2}(ab)} \alpha_i f_i(z).$$

- By comparing the n-th Fourier coefficients on both sides and simplifying we have

$$W_{a,b}^{1,k-1}(n) = \frac{B_k}{2k}\sigma(n/a) + \left(\frac{1}{24} - \frac{n}{2ka}\right)\sigma_{k-1}(n/b) - \frac{B_k}{4ak^2}\sum_{i=1}^{\lambda_{k+2}(ab)} \alpha_i a_i(n),$$

where B_k is the k-th Bernoulli number.

Application 2: Convolution sums

Application 2: Convolution sums

- For $gcd(a,b)=1$, the function

$$\frac{6a}{b}DE_2(az)+\frac{6b}{a}DE_2(bz)-E_2(az)E_2(bz)\in M_4(ab)$$

- Let's denote a basis of $M_4(ab)$ by $g_i(z), 1\leq i\leq \lambda_4(ab)$ whose n-th Fourier coefficients are given by $b_i(n)$.

- We obtain the convolution sum $W_{a,b}(n)$ by using a basis for the space $M_4(ab)$.

$$W_{a,b}(n)=\frac{1}{24}\left(1-\frac{6n}{b}\right)\sigma(n/a)+\frac{1}{24}\left(1-\frac{6n}{a}\right)\sigma(n/b)-\frac{1}{576}\sum_{i=1}^{\lambda_4(ab)}\beta_i b_i(n),$$

for some $\beta(i)$ some constants.

Application 3: Convolution sums

Application 3: Convolution sums

- Apply Ramanujan-Serre derivative $\delta_{k,a,b}$ on $E_k(z)$: For $gcd(a,b)=1$,

$$bDE_k(bz) - \frac{k}{12} aE_2(az)E_k(bz) \in M_{k+2}(ab)$$

- Apply $\delta_{k+2,a,1}$ to the above modular form to obtain

$$D(bDE_k(bz) - \frac{k}{12} aE_2(z)E_k(bz)) - \frac{k+2}{12} aE_2(az)(bDE_k(bz) - \frac{k}{12} aE_2(z)E_k(bz))$$

  a modular form in $M_{k+4}(ab)$.

- Use $E_2^2(az) = \frac{12}{a} DE_2(az) + E_4(az)$ and $E_4(az)E_k(bz) \in M_{k+4}(ab)$ to conclude that

$$b^2 D^2 E_k(bz) + \frac{ka^2}{12}(k+1)DE_2(az)E_k(bz) - \frac{ab(k+1)}{6} E_2(az)DE_k(bz) \in M_{k+4}(ab)$$

Application 3: Convolution sums

Application 3: Convolution sums

- Now considering $h_i(z), 1 \leq i \leq \lambda_{k+4}(ab)$ a basis for the space $M_{k+4}(ab)$ with n-th coefficient $c_i(n)$ we prove the following:

$$\sum_{\substack{l,m \in \mathbb{N} \\ al+bm=n}} l\sigma(l)\sigma_{k-1}(m) = \frac{6n^2 - a(k+1)n}{12a^2(k+1)(k+2)}\sigma_{k-1}(n/b) + \frac{B_k}{2a(k+2)} n\sigma(n/a)$$

$$+\frac{2n}{a(k+2)} W_{a,b}^{1,k-1}(n) + \frac{B_k}{4a^2k(k+1)(k+2)} \sum_{i=1}^{\lambda_{k+4}(ab)} \gamma_i c_i(n)$$

where B_k is the k-th Bernoulli number, $\gamma_i \in \mathbb{C}$ are constants, $\lambda_{k+4}(ab)$ is the dimension of $M_{k+4}(ab)$

Application 4: Convolution sums

Application 4: Convolution sums

- For gcd $(a, b) = 1$, we have

$$\frac{6a}{b} DE_2(az) + \frac{6b}{a} DE_2(bz) - E_2(az)E_2(bz) \in M_4(ab)$$

- Apply the operator $\delta_{4,a,1}$ to above modular form to get

$$D\left(\frac{6a}{b} DE_2(az) + \frac{6b}{a} DE_2(bz) - E_2(az)E_2(bz)\right)$$

$$-\frac{a}{3}E_2(az)\left(\frac{6a}{b} DE_2(az) + \frac{6b}{a} DE_2(bz) - E_2(az)E_2(bz)\right)$$

  to get a modular form of weight 6.

- Consider a basis $F_i(z), 1 \leq i \leq \lambda_6(ab)$ for the space $M_6(ab)$ with n-th coefficient $A_i(n)$.

Application 4: Convolution sums

Application 4: Convolution sums

- We have the following :

$$\sum_{\substack{l,m\in\mathbb{N}\\ al+bm=n}} l\sigma(l)\sigma(m) = \frac{1}{144}\left(\frac{6}{b}n^2+3n-\frac{2an}{b}\right)\sigma(n/a)+\frac{1}{144a}\left(\frac{6}{a}n^2+3n-\frac{a}{3}\right)\sigma(n/b)$$

$$-\frac{5}{216}\sigma_3(n/a)+\frac{n}{2a}W_{a,b}(n)+\frac{5}{9}W_{b,a}^{1,3}(n)+\frac{a}{3b}\sum_{\substack{l,m\in\mathbb{N}\\ al+bm=n/a}} l\sigma(l)\sigma(m)+\frac{1}{3456a}\sum_{i=1}^{\lambda_6(ab)}\delta_i A_i(n)$$

$\delta_i \in \mathbb{C}$ are constants, $\lambda_6(ab)$ is the dimension of $M_6(ab)$

- The sum of the type

$$W_{a,b}^{e:r,s} := \sum_{\substack{l,m\in\mathbb{N}\\ al+bm=n}} l^e\sigma_r(l)\sigma_s(m)$$

can be evaluated.

Examples

Examples

- $W_{a,b}(n)$ formula two cases $(a,b) = (1,15),(3,5)$ are given below.

$$W_{15}(n) = \frac{1}{624}\sigma_3(n) + \frac{3}{208}\sigma_3(n/3) + \frac{25}{624}\sigma_3(n/5) + \frac{75}{208}\sigma_3(n/15) + \frac{(5-2n)}{120}\sigma(n) + \frac{(1-6n)}{24}\sigma(n/15) - \frac{1}{455}\tau_{4,5}(n) - \frac{9}{455}\tau_{4,5}(n/3) - \frac{1}{80}\tau_{4,15;1}(n) - \frac{1}{84}\tau_{4,15;2}(n),$$

$$W_{3,5}(n) = \frac{1}{624}\sigma_3(n) + \frac{3}{208}\sigma_3(n/3) + \frac{25}{624}\sigma_3(n/5) + \frac{75}{208}\sigma_3(n/15) + \frac{(5-6n)}{120}\sigma(n/3) + \frac{(1-2n)}{24}\sigma(n/5) - \frac{1}{455}\tau_{4,5}(n) - \frac{9}{455}\tau_{4,5}(n/3) + \frac{1}{80}\tau_{4,15;1}(n). - \frac{1}{84}\tau_{4,15;2}(n)$$

$\tau_{k,N,j}$ denotes the nth Fourier coefficient of $j-$th newform in $S_k^{new}(N)$.

Examples

Examples

- $W_{a,b}^{1,k-1}$: Formula for $k = 4, (a,b) = (1,3), (3,1), (1,6), (6,1)$ are given below.

$$W_{1,3}^{1,3}(n) = \frac{1}{1040}\sigma_5(n) + \frac{9}{104}\sigma_5(n/3) + \frac{(1-3n)}{24}\sigma_3(n/3) - \frac{1}{240}\sigma(n) + \frac{1}{312}\tau_{6,3}(n),$$

$$W_{3,1}^{1,3}(n) = \frac{1}{104}\sigma_5(n) + \frac{81}{1040}\sigma_5(n/3) + \frac{(1-n)}{24}\sigma_3(n) - \frac{1}{240}\sigma(n/3) - \frac{1}{104}\tau_{6,3}(n),$$

$$W_{1,6}^{1,3}(n) = \frac{1}{21840}\sigma_5(n) + \frac{1}{1092}\sigma_5(n/2) + \frac{3}{728}\sigma_5(n/3) + \frac{15}{182}\sigma_5(n/6) + \frac{(1-3n)}{24}\sigma_3(n/6$$
$$- \frac{1}{240}\sigma(n) + \frac{1}{468}\tau_{6,3}(n) + \frac{7}{468}\tau_{6,3}(n/2) + \frac{1}{504}\tau_{6,6}(n),$$

$$W_{6,1}^{1,3}(n) = \frac{5}{2184}\sigma_5(n) + \frac{2}{273}\sigma_5(n/2) + \frac{27}{1456}\sigma_5(n/3) + \frac{27}{455}\sigma_5(n/6) + \frac{(2-n)}{48}\sigma_3(n)$$
$$- \frac{1}{240}\sigma(n/6) - \frac{7}{624}\tau_{6,3}(n) - \frac{4}{39}\tau_{6,3}(n/2) - \frac{1}{84}\tau_{6,6}(n).$$

Examples

Examples

- $\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}} l\sigma(l)\sigma(m) = \frac{n}{24}\sigma_3(n) + \frac{n}{6}\sigma_3(n/2) - \frac{(2n^2-n)}{24}\sigma(n) - \frac{n^2}{12}\sigma(n/2),$
- $\sum_{\substack{l,m\in\mathbb{N}\\ l+3m=n}} l\sigma(l)\sigma(m) = \frac{n}{48}\sigma_3(n) + \frac{3n}{16}\sigma_3(n/3) - \frac{(4n^2-3n)}{72}\sigma(n) - \frac{n^2}{12}\sigma(n/3) - \frac{1}{144}\tau_{6,3}(n).$
- $\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}} l\sigma(l)\sigma_3(m) = \frac{n}{720}\sigma_5(n) + \frac{n}{36}\sigma_5(n/2) - \frac{n^2}{40}\sigma_3(n/2) - \frac{n}{240}\sigma(n) + \frac{1}{360}\tau_{8,2}(n),$
- $\sum_{\substack{l,m\in\mathbb{N}\\ l+3m=n}} l\sigma(l)\sigma_3(m) = \frac{n}{3120}\sigma_5(n) + \frac{3n}{104}\sigma_5(n/3) - \frac{n^2}{40}\sigma_3(n/3) - \frac{n}{240}\sigma(n) + \frac{n}{936}\tau_{6,3}(n),$

- As one major of the major applications of these convolution sums is finding formulas for the number of representation of an integer by certain quadratic forms.
- Consider the quadratic forms in $2k$ variables given by

$$Q_{2k} : (x_1^2 + x_1x_2 + x_2^2 + \cdots + x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2)$$

and $s_{2k}(n)$ denote the number of representations of a non-negative integer n by Q_{2k}.
- By elementary evaluations, it is know that

$$s_4(n) = 12\sigma(n) - 36\sigma(n/3).$$

- Now

$$s_8(n) = \sum_{\substack{a,b\in\mathbb{N}\cup\{0\}\\ a+b=n}} s_4(a)s_4(b)$$

Using the formula of $s_4(n)$, we have

$$s_8(n) = 24\sigma(n) - 72\sigma(n/3) + 144W_1(n) - 864W_3(n) + 1296W_1(n/3)$$

Using the formulas for $W_a(n)$ in the above, we get

$$s_8(n) = 24\sigma_3(n) + 216\sigma_3(n/3).$$

- One can consider $Q_{8,a,b} : aQ_4 \oplus bQ_4$ and proceeding as above we can find a formula for the number of representations of a positive integer n by $Q_{8,a,b}$ in terms of the convolution sums $W_{a,b}(n)$, $W_a(n)$, $W_b(n)$.

- In a similar way one can consider $Q_{16,a,b} : aQ_8 \oplus bQ_8$ and find a formula for the number of representations of a positive integer in terms of the convolution sums $W^{3,3}_{a,b}(n)$, $W^{3,3}_{a,3b}(n)$.

- Formula for representation of an integer as sum of odd number of squares using convolutions sum (with Shivansh Pandey, 2022):

$$r_{11}(n^2) = \sum_{d|n} \mu(d) d^4 \left(\frac{-4}{d}\right) \left[\frac{330}{31}\sigma_9\left(\frac{n}{d}\right) - \frac{10560}{31}\sigma_9\left(\frac{n}{2d}\right)\right.$$

$$\left. + \frac{352}{31}\left(\sigma_5\left(\frac{n}{d}\right) - \sigma_5\left(\frac{n}{2d}\right)\right) + \frac{5632}{31} W^*_{4,6}\left(\frac{n}{d}\right)\right].$$

- $r_{11}(p^2) = \frac{330}{31}\sigma_9(p) - 22(-1)^{\frac{p-1}{2}} p^4 + \frac{352}{31}\sigma_5(p) + \frac{5632}{31} W^*_{4,6}(p).$

- In a recent work, we extended our method to evaluate triple convolution sums of the following type: For r, s, t are odd integers ≥ 1,

$$W_{a,b,c}^{r,s,t}(n) = \sum_{\substack{l_1,l_2,l_3\in\mathbb{N}\\ al_1+bl_2+cl_3=n}} \sigma_r(l_1)\sigma_s(l_2)\sigma_t(l_3)$$

- For positive integers n, d_1, d_2, d_3 such that $\gcd(d_1, d_2, d_3) = 1$, we have

$$W_{d_1,d_2,d_3}^{1,1,1}(n) = -\frac{1}{24^2}\sigma\left(\frac{n}{d_1}\right) + \frac{1}{24^2}\left(-\frac{18n^2}{d_1d_3} + \frac{3n}{d_1} + \frac{6n}{d_3} - 1\right)\sigma\left(\frac{n}{d_2}\right) + \frac{1}{24}\left(1 - \frac{6n}{d_3}\right)W_{d_1,d_2}(n)$$

$$+\frac{1}{24^2}\left(-\frac{18n^2}{d_1d_2} + \frac{3n}{d_1} + \frac{6n}{d_2} - 1\right)\sigma\left(\frac{n}{d_3}\right) + \frac{1}{24}\left(1 - \frac{3n}{d_1}\right)W_{d_2,d_3}(n)$$

$$+\frac{1}{24}\left(1 - \frac{6n}{d_2}\right)W_{d_1,d_3}(n) + \frac{d_1}{4d_3}W_{\widetilde{d}_1,d_2}(n) + \frac{d_1}{4d_2}W_{\widetilde{d}_1,d_3}(n) + \frac{1}{24^3}\sum_{i=1}^{\lambda_6(d)} \alpha_i a_{f_i}(n),$$

where $a_{f_i}(n)$ denotes the n-th Fourier coefficient of the i-th basis element $f_i(z)$, $1 \leq i \leq \lambda_6(d)$ of a basis for the vector space $M_6(d)$, $d = \mathrm{lcm}(d_1, d_2, d_3)$.

- Evaluated triple convolutions $W_{d_1,d_2,d_3}^{1,1,k-1}(n)$, $W_{d_1,d_2,d_3}^{1,k_2-1,k_3-1}(n)$, $W_{d_1,d_2,d_3}^{k_1,k_2-1,k_3-1}(n)$.

References

References

References

- B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, A simple extension of Ramanujan-Serre derivative map and some applications, The Ramanujan Journal, 61 (2023), 1379-1410.

- B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, Explicit evaluation of triple convolution sums of the divisor functions, International Journal Number Theory, 20 (2024), 4, 1073-1098

Thank You!