The Non-vanishing of Traces of Hecke Operators

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Based on Joint Work with Liubomir Chiriac and Daphne Kurzenhauser

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We will focus mostly on 1 in this presentation.

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For all positive integers N and $m \geq 2$ coprime to N, the function

$$R(\Gamma_0(N), m; x) := \sum_{k=1}^{\infty} \mathrm{Tr}\, T_m(2k, \Gamma_0(N)) x^{k-1}$$

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This means that Tr $T_m(2k, \Gamma_0(N))$ is always governed by a linear recurrence relation in k (in fact it gives an explicit recurrence).
Once one has a linear recurrence for Tr $T_m(2k, \Gamma_0(N))$ in k, one can study the sequence with p-adic methods, asymptotics, etc.

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Rouse's approach involves the notion of "projective equivalence" which allows one to reduce the problem to checking that only finitely many recurrences are nonvanishing.

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Strategy.

1. Use the Eichler-Selberg Trace Formula to find a convenient combinatorial formula for $\mathrm{Tr}\, T_3(2k)$.
2. Exhibit a linear recurrence for $\mathrm{Tr}\, T_3(2k)$.
3. Use p-adic methods to show that this linear recurrence only vanishes at the desired values of k.

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For $\mathrm{SL}_2(\mathbb{Z})$, one formulation of the Eichler-Selberg Trace Formula is given by Zagier:

$$\mathrm{Tr}\ T_m(2k) = -\frac{1}{2}\sum_{t^2 \leq 4m} P_{2k}(t,m)H(4m-t^2) - \frac{1}{2}\sum_{d|m} \min(d, m/d)^{2k-1}$$

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This means

$$\text{Tr } T_3(2k) = -\frac{2}{3}P_{2k}(0,3) - P_{2k}(1,3) - P_{2k}(2,3) - \frac{1}{3}P_{2k}(3,3) - 1.$$

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Furthermore, it's straightforward to show that

$$P_{2k}(t,m) = \sum_{j=0}^{k-1}(-1)^j \binom{2k-2-j}{j} m^j t^{2k-2-2j}.$$

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$$S_3 = 2 \cdot (-3)^{k-2} + \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} 3^{2k-3-j}.$$

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Proposition

Let $\{a_n\}_{n\geq 0}$ *and* $\{b_n\}_{n\geq 0}$ *be given by*

$$a_n = -5a_{n-1} - 9a_{n-2} \textit{ for } n \geq 2 \quad a_0 = 1, \quad a_1 = -2$$

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Then for all $k \geq 2$, *we have*

$$\mathrm{Tr}\, T_3(2k) = \begin{cases} -1 - a_{k-1} - b_{k-1} & \text{if } k \equiv 2,5 \pmod 6 \\ -1 - a_{k-1} - b_{k-1} - 3^{k-1} & \text{if } k \equiv 1,3 \pmod 6 \\ -1 - a_{k-1} - b_{k-1} + 3^{k-1} & \text{if } k \equiv 0,4 \pmod 6 \end{cases}$$

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Some algebra reduces this to finding an explicit formula for

$$\sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} 3^{n-j}$$

which can be done with the negative binomial series and induction. □

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p-adic Methods

Proposition (Mignotte and Tzanakis, 1991)

Suppose that $\mathcal{M}$ is a finite set of solutions $m \in \mathbb{Z}$ to the equation $u_n = c$, where $c \not\equiv 0 \pmod{p}$ or $c = 0$.

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This result allows us to verify that a set of solutions $\mathcal{M}$ in fact contains *all* solutions to $u_n = c$.

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$$u_n = \alpha_1 \omega_1^n + \ldots + \alpha_k \omega_k^n$$

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for some $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$. Using this formula, it makes sense to consider u_n for any $n \in \mathbb{Z}$.

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Recall that our explicit recurrence for $\text{Tr } T_3(2k)$ involved the sequences

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We let $u_n = a_n + b_n$.

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The first few values of each sequence are:

n	-1	0	1	2	3	4	5	6
u_n	$-2/3$	2	$-\mathbf{1}$	-10	26	$-\mathbf{1}$	-10	-730
$u_n + 3^n$	$-1/3$	3	2	$-\mathbf{1}$	53	80	233	$-\mathbf{1}$
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The roots of g (reduced modulo 59) turn out to be quadratic residues, so

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- Here $u_n - 3^n = -1$ has a negative solution. If one takes the system of residues $\mathcal{P} = \{-1, \ldots, 27\}$ to accommodate for the negative solution, $(p, S, A) = (59, 29, 1)$ will work.

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My current research project is extending the methods of Rouse to show the non-vanishing of T_3 in all levels.

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Questions?