A general approach to the modularity of hypergeometric Galois representations

36th Automorphic Forms Workshop

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Hypergeometric functions

Definition

Given multisets of rationals $\alpha = \{r_1, \ldots, r_n\}$ and $\beta = \{1, q_1, \ldots, q_{n-1}\}$ we define the generalized hypergeometric series ${}_nF_{n-1}(\alpha, \beta)$ by

$$ {}_nF_{n-1}\left[\begin{matrix} r_1 & r_2 & \cdots & r_n \\ & q_1 & \cdots & q_{n-1} \end{matrix} \,\middle|\, z \right] := \sum_{k=0}^{\infty} \frac{(r_1)_k (r_2)_k \cdots (r_n)_k}{(q_1)_k \cdots (q_{n-1})_k} \frac{z^k}{k!}, $$

where $(a)_k$ denotes the Pochhammer symbol

$$ (a)_0 := 1 \quad \text{and} \quad (a)_k := a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \text{ for } k \geq 1. $$

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$HD = \{\alpha, \beta\}$—*hypergeometric datum*.
Throughout, $r_i \in (0,1)$ and $q_i \in (1/2, 3/2)$.

Finite field hypergeometric functions

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Definition (Greene, McCarthy)

Let $\alpha = \{r_1, r_2, \ldots, r_n\}$ and $\beta = \{1, q_2, \ldots, q_n\}$. Set $M = \mathrm{lcd}(\{r_i, q_i\})$. Let ω be a generator of the character group $\widehat{\mathbb{F}_q^\times}$. For a fixed prime power $q = p^r$ which is congruent to 1 modulo M.

$$H_q(\alpha, \beta; \lambda; \omega) := \frac{1}{1-q} \sum_{k=0}^{q-2} \omega^k((-1)^n\lambda) \prod_{j=1}^{n} \frac{\mathfrak{g}\left(\omega^{(q-1)r_j+k}\right)}{\mathfrak{g}\left(\omega^{(q-1)r_j}\right)} \frac{\mathfrak{g}\left(\bar{\omega}^{(q-1)q_j+k}\right)}{\mathfrak{g}\left(\bar{\omega}^{(q-1)q_j}\right)}.$$

Geometric motivation

Legendre family of elliptic curves:

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Periods if $0 < \lambda < 1$:

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; \lambda\right] = \frac{1}{\pi}\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} $$

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Point counts (Koike, 1992):

$$\phi(-1)H_p\left(\left\{\frac{1}{2},\frac{1}{2}\right\},\{1,1\};\lambda\right) = \#E_\lambda(\mathbb{F}_p) - p - 1.$$

Hypergeometric Galois representations

Theorem (Katz)

Let ℓ be a prime. Given a primitive hypergeometric datum $HD = \{\alpha, \beta\}$ with $n = |\alpha| = |\beta|$, $M = \mathrm{lcd}(\alpha, \beta)$ and $\lambda \in \mathbb{Z}[\zeta_M, 1/M] \setminus \{0\}$, there is a continuous ℓ-adic Galois representation

$$\rho_{HD,\ell} : G_{\mathbb{Q}(\zeta_M)} \to \mathrm{GL}_{n-\mathbb{1}_{\lambda=1}}(\mathbb{C}_\ell)$$

If this is extendable to a representation $\widetilde{\rho_{HD,\ell}}$ of the full absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then we have

$$\mathrm{Tr}\widetilde{\rho_{HD,\ell}}(\mathrm{Frob}_p) =^* H_p(\alpha, \beta, \lambda, \omega_p).$$

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These are nice (unramified almost everywhere, often semi-simple, etc.)

Modularity of hypergeometric representations

Theorem (A., Grove, Long, Tu, 2024)

Let $n = 3$ or 4. Assume $HD^\flat = \{\alpha^\flat = \{r_1, \cdots, r_{n-1}\}, \beta^\flat = \{1, \cdots, 1\}\}$ is a primitive datum, $r_n, q_n \in \mathbb{Q}$, $t = t(\tau) = t(q^{1/\mu}) = C_1 q^{1/\mu} + \text{higher order terms} \in \mathbb{Z}[[q^{1/\mu}]]$ is a modular function such that

$$f(q) := C_1^{-r_n} \cdot t(q)^{r_n}(1 - t(q))^{q_n - r_n - 1} {}_{n-1}F_{n-2}(\alpha^\flat, \beta^\flat; t(q)) \frac{dt(q)}{t(q)dq}$$

is a congruence weight n holomorphic cuspform. Set $HD = \{\{r_n\} \cup \alpha^\flat, \{q_n\} \cup \beta^\flat\}$, $K = \mathbb{Q}(\zeta_{M(HD)})$, and let χ be a character of G_K defined explicitly as an e^{th} residue symbol. If $\chi \otimes \rho_{\{HD;1\}}$ can be extended to $G_\mathbb{Q}$, then there exists an explicit Hecke eigenform $f^\sharp$ such that $\rho_{f^\sharp}|_{G_K}$ is isomorphic to the primitive part of $\chi \otimes \rho_{\{HD;1\}}$, where $\rho_{f^\sharp}$ denotes the Deligne representation of $G_\mathbb{Q}$ associated with $f^\sharp$.

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Oversimplifying, we need to identify H_p values with $a_p(f^\sharp)$.

Identifying the target modular form

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Proposition (AGLT)

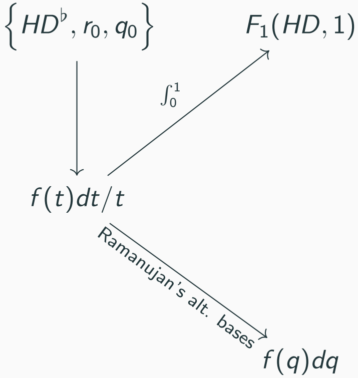
There are 167 pairs (r_3, q_3) for which the integrand

$$f(t) = 16^{-r_n} \cdot t^{r_3}(1-t)^{q_3-r_3-1}{}_2F_1(\alpha^\flat, \beta^\flat; t)\frac{dt}{t}$$

at $t = \lambda(\tau)$ is a weight 3 holomorphic cusp form. For each of these cases,

$$f(\lambda(\tau)) = \frac{\eta\left(\frac{\tau}{2}\right)^{16q_3-8r_3-12}\eta(2\tau)^{8q_3+8r_3-12}}{2\eta(\tau)^{24q_3-30}}.$$

Sketch of the proof



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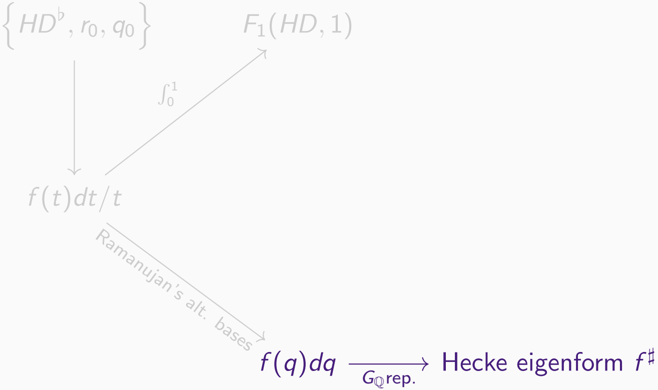

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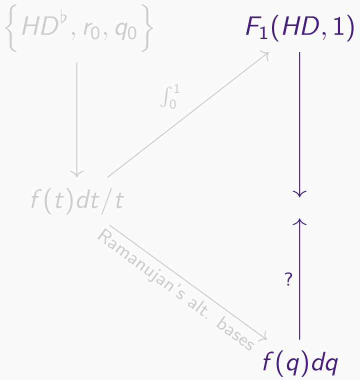

Commutative formal group law

The integral relation between our hypergeometric function and our modular form, along with Commutative Formal Group Law, gives

$$ {}_nF_{n-1}\{\alpha, \beta; 1\}_{p-1} \equiv^* a_p(f(q)) \pmod{p}. $$

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$\alpha = \alpha^\flat \cup \{r_n\}, \beta = \beta^\flat \cup \{q_n\}$, and the subscript denotes truncation at $p-1$.

Gross–Koblitz formula

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$$\omega_p \in \widehat{\mathbb{F}_p^\times} : \omega_p(a) \equiv a \pmod{p}.$$

$$\pi_p : \text{ fixed root of } x^{p-1} + p = 0 \text{ in } \mathbb{C}_p.$$

$$\Gamma_p(x) : \mathbb{Z}_p \to \mathbb{Z}_p^\times : \qquad \Gamma_p(n) = (-1)^n \prod_{0<i<n, p\nmid i} i \qquad 0 < n \in \mathbb{Z}.$$

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$$\mathfrak{g}(\omega_p^{-r}) = -\pi_p^r \Gamma_p\left(\frac{r}{p-1}\right).$$

H_p congruences

Via Gross–Koblitz, there are congruences of the form

$$H_p(\overline{\alpha}, \overline{\beta}; 1) \equiv {}_nF_{n-1}(\alpha, \beta; 1)_{p-1} \equiv^* a_p(f_{HD}) \pmod{p}.$$

$$\overline{\alpha} = \{1 - r : r \in \alpha\} \qquad \overline{\beta} = \{1 - q + \lfloor q \rfloor : q \in \beta\}$$

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Galois extendability ensures the left-hand side is integral, so we only need a congruence modulo p^2 to get an equality.

Supercongruences

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$$H_p(\overline{\alpha}, \overline{\beta}; 1) \equiv {}_nF_{n-1}(\alpha, \beta; 1)_{p-1} + E_{\mathrm{GK}}(\alpha, \beta; 1)p \pmod{p^2}$$

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Theorem (AGLT)

For any positive integer n, let $\alpha = \{r_1, r_2, \ldots, r_n\}, \beta = \{q_1, \ldots, q_n\}$ satisfy

1. $0 < r_1 \leq \cdots \leq r_n < 1, 0 < q_1 \leq \cdots \leq q_{n-2} \leq q_{n-1} = q_n = 1$, *and* $q_i > r_{i+2}$,
2. $\gamma(HD) = -1 + \sum_{i=1}^{n}(q_i - r_i) < 1$
3. $p \equiv 1 \pmod{\mathrm{lcd}(\alpha,\beta)}$, $p > \#\{1 \leq i \leq n : q_i \neq 1\}$, $F(\alpha,\beta;1)_{p-1} \not\equiv 0 \pmod{p}$,

$$E_{\mathrm{GK}} \equiv E_{\mathrm{Dwork}} \equiv 0 \pmod{p}.$$

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Theorem (AGLT)

For any positive integer n, let $\alpha = \{r_1, r_2, \ldots, r_n\}, \beta = \{q_1, \ldots, q_n\}$ *satisfy*

1. $0 < r_1 \leq \cdots \leq r_n < 1, 0 < q_1 \leq \cdots \leq q_{n-2} \leq q_{n-1} = q_n = 1$, *and* $q_i > r_{i+2}$,
2. $\gamma(HD) = -1 + \sum_{i=1}^{n}(q_i - r_i) < 1$
3. $p \equiv 1 \pmod{\mathrm{lcd}(\alpha, \beta)}$, $p > \#\{1 \leq i \leq n : q_i \neq 1\}$, $F(\alpha, \beta; 1)_{p-1} \not\equiv 0 \pmod{p}$,

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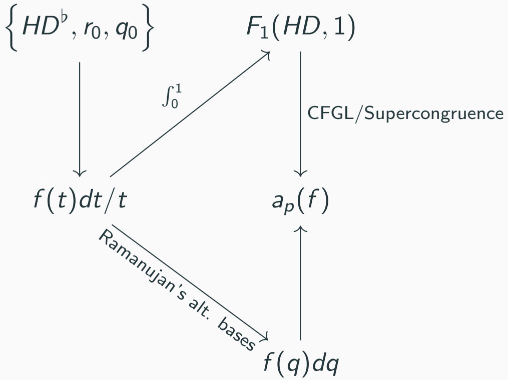
Sketch of the proof



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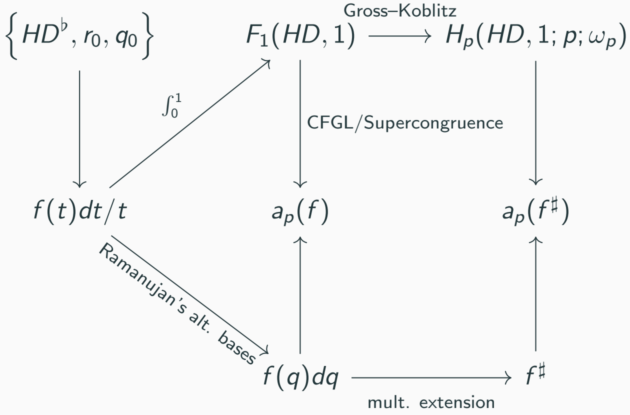

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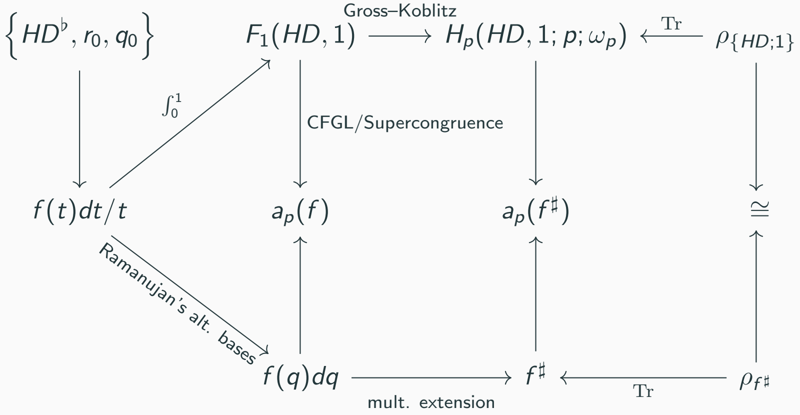

Example

$$HD_2 = \left\{\left\{\frac{1}{2}, \frac{1}{2}\right\}, \{1, 1\}\right\}$$
$$HD_{2,j} := \left\{\left\{\frac{j}{8}, \frac{1}{2}, \frac{1}{2}\right\}, \{1, 1, 1\}\right\}.$$

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Setting $t = \lambda(\tau)$ in $f_j(t)$ and applying the theorem gives

$$H_p(HD; 1) = a_p(f_j(q)),$$

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$j = 2, 4$ recover results of McCarthy–Papanikolas and Ahlgren

$j = 1$ resolves a conjecture of Dawsey–McCarthy.

One more application—computing L-values

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Similarly, we can relate $L(f_{32.3.c.a}, 1)$ with $HD = \left\{ \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{4} \right\}, \left\{ 1, 1, \frac{3}{4} \right\} \right\}$.

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$$L(f_{64.3.c.b}, 1) = \sqrt{-2} L(f_{32.3.c.a}, 1).$$

Thank you!