Mass Equidistribution: Cocompact vs. Non-Cocompact Surfaces

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Quaternion Algebras

- We let $A = \left(\frac{a,b}{\mathbb{Q}}\right)$ be a quaternion algebra that is split over $\mathbb{R}$ (i.e. $A \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$).
- We may explicitly write down this isomorphism, letting $\alpha = t + xi + yj + zij = \xi + \eta j$ where $\xi, \eta \in \mathbb{Q}(\sqrt{a}) \subset \mathbb{R}$, by

$$\alpha \mapsto \begin{bmatrix} \bar{\xi} & \eta \\ b\eta & \xi \end{bmatrix}.$$

- Let $R \subset A$ be a maximal order, then under the isomorphism above we will have that $R(1) = \{\alpha \in R : N_{red}(\alpha) = 1\}$ will be a subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Example

If A is split over $\mathbb{Q}$, then $A \cong M_2(\mathbb{Q})$. Then a maximal order $R \subset A$ will be isomorphic to $M_2(\mathbb{Z}) \subset M_2(\mathbb{Q})$. Hence, $R(1) \cong \mathrm{SL}_2(\mathbb{Z})$.

Modular Forms

- There is a strict dichotomy between the algebra of A and the geometry of the surface $R(1)\backslash\mathbb{H}$:
  - If A is split over $\mathbb{Q}$, then $R(1)\backslash\mathbb{H}$ is not cocompact.
  - If A is non-split over $\mathbb{Q}$, then $R(1)\backslash\mathbb{H}$ is a cocompact.
- A modular form of weight k on $R(1)\backslash\mathbb{H}$ is a function $f:\mathbb{H}\to\mathbb{C}$ such that for $\gamma\in R(1)$:

$$f(\gamma z) = (cz+d)^k f(z)$$

- Let $R(n) = \{\alpha\in R : N_{red}(\alpha) = n\}$, we define the Hecke operators by

$$(T_n f)(z) = \sum_{\gamma\in R(1)\backslash R(n)} f(\gamma z).$$

- Letting f be a joint eigenfunction of all Hecke operators, we shall denote $\lambda_f(n)$ the n-th Hecke eigenvalue.
- If A is split, then f will have a Fourier expansion of

$$f(z) = \sum_{n\geq 1}\lambda_f(n) n^{\frac{k-1}{2}} e(nz)$$

Quantum Ergodicity

Conjecture (Quantum Unique Ergodicity (QUE))

Let (M, g) be a smooth Riemannian manifold, there is an associated probability measure μ obtained from g. Let $-\Delta_g$ denote the Laplace-Beltrami operator, and pick ϕ_j an orthonomal basis of $L^2(M)$ of eigenfunctions $-\Delta_g \phi_j = \lambda_j \phi_j$ with $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$. Then for any smooth test function $\psi \in C^\infty(M)$, we have that

$$\lim_{j \to \infty} \int_M \psi(x) |\phi_j(x)|^2 d\mu(x) = \int_M \psi(x) d\mu(x)$$

Theorem (Hassell 2010)

QUE is false.

Arithmetic QUE

Conjecture (Rudnick and Sarnak 1994)

QUE holds when M is a compact manifold of negative curvature.

- QUE is still far from being solved.
- Arithmetic QUE is the statement of QUE for arithmetic surfaces.

Theorem (Lindenstrauss 2006)

QUE is true for compact arithmetic surfaces.

Theorem (Soundararajan 2010)

QUE is true for Hecke-Maass cusp forms on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

Mass Equidistribution

- There is a general philosophy that when there is a statement about Maass forms there should be an analogous statement about modular forms with the weight of the form corresponding to the notion of the Laplace eigenvalue.

Conjecture (Mass Equidistribution Conjecture Luo Sarnak 2003)

Let f_k be a sequence of Hecke cusp forms, then for any test function $\psi \in C^\infty(M)$, then

$$\lim_{k\to\infty} \int_M \psi(x)|f_k|^2 d\mu(x) = \int_M \psi(x) d\mu(x).$$

- The condition of the f_k being Hecke cusp forms is necessary for the statement of the theorem to be true.

Theorem (Holowinsky Soundararajan 2008)

Mass Equidistribution is true for Hecke cusp forms on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

Shifted Convolution Sums

Theorem (Luo-Sarnak 2003)

The mass equidistribution conjecture for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ *is equivalent to the following statement about shifted convolution sums. For each* $m \geq 0$, *and* $\psi \in C_0^\infty(0,\infty)$, *we have that*

$$\lim_{k\to\infty} \frac{2\pi^2}{(k-1)L(1,\mathrm{Sym}^2(f))} \sum_{r\geq 1} \lambda_f(r)\lambda_f(r+m)\psi\left(\frac{k-1}{4\pi(r+\frac{m}{2})}\right)$$

$$= \frac{3}{\pi}\delta_{m,0}\int_0^\infty \psi(y)\frac{dy}{y^2}$$

- The proof of the equivalence uses incomplete Poincaré series

$$P_{\psi,m}(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\mathrm{Im}(\gamma z))e(m\,\mathrm{Re}(\gamma z))$$

Hueristic Towards Cocompact Case

- Using Hecke multiplicatively and Möbius inversion, mass equidistribution is concerned with convolution sums of the form:

$$\frac{1}{k}\sum_{r\geq 1}\lambda_f(q(r))\psi\left(\frac{r}{k}\right)$$

where $q(r) = r(r+m)$.
- One might hope that for the non-split case, we can reformulate the shifted convolution sums method by replacing the reducible quadratic $q(r)$ with an irreducible quadratic.

Theorem (Nelson 2022)

If for every irreducible quadratic integer polynomial $q \in \mathbb{Q}[x]$, and $\psi \in C_0^\infty(0,\infty)$, QUE for cocompact surfaces follow from

$$\lim_{k\to\infty}\frac{1}{k}\sum_{r\geq 1}\frac{\lambda_f(|q(r)|)}{L(1,\mathrm{Sym}^2(f))}\psi\left(\frac{r}{k}\right) = 0$$

Work in Progress

Theorem (C. 2024 +)

Let $q(x)$ be an irreducible quadratic polynomial, $H_k(\Gamma_0(N))$ an orthogonal basis of Hecke cusp forms for $S_k(\Gamma_0(N))$. Let X, θ, ϵ be some parameters where $1 \ll X \ll K$, $\frac{1}{3} < \theta < 1$, and $\epsilon > 0$. Assume that $\psi \in C_0^\infty(0, \infty)$, then:

$$\frac{1}{X} \sum_{|k-K|<K^\theta} \sum_{f \in H_k(\Gamma_0(N))} \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \left| \sum_{r \geq 1} \lambda_f(|q(r)|) \psi\left(\frac{r}{X}\right) \right|^2 \ll K^{1+\theta+\epsilon}.$$

Thank you

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