Fibers of Projected Richardson Varieties

Travis Grigsby

Oklahoma State University

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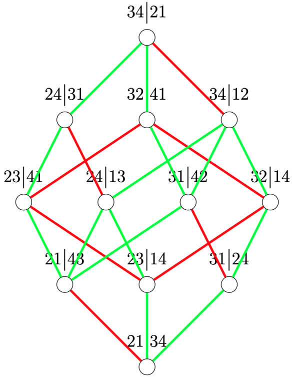
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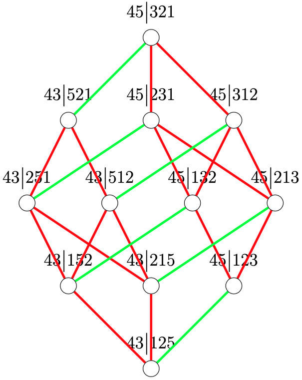

45|321
43|521
45|231
45|312
43|251
43|512
45|132
45|213
43|152
43|215
45|123
43|125

Background and Notation:

$G = SL_n(\mathbb{C})$ (G a reductive algebraic group over $\mathbb{C}$.)

$$T = \begin{bmatrix} * & & & & \\ & * & & 0 & \\ & & \ddots & & \\ & 0 & & * & \\ & & & & * \end{bmatrix}, B = \begin{bmatrix} * & * & \cdots & * & * \\ & * & \cdots & * & * \\ & & \ddots & \vdots & \vdots \\ & 0 & & * & * \\ & & & & * \end{bmatrix}$$

Define the Weyl group $W = \textbf{Norm}(T)/T \cong S_n = \{$ Permutation matrices $\}$.

Right action by B on G preserves the span of the iterative spans of column vectors of matrices.

$G/B = \{V_\bullet := V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n : \textbf{dim} V_i = i\}$

G/B is called the **flag variety**, our ambient space.

Background and Notation:

Let s_i denote the permutation swapping the i^{th} and $(i+1)^{\text{th}}$ columns of a matrix.

$W = S_n = \langle s_i \rangle$; generated by simple reflections

$G/B = \bigsqcup_{w \in S_n} BwB/B$; the Bruhat decomposition

Background and Notation:

$$P = \begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & 0 & & * & * \\ & & & & * \end{bmatrix} \quad ; \quad \text{Parabolic Subgroup of } G$$

Only some of the spans of columns are preserved.

Example: $G = SL_5(\mathbb{C})$ and $P = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & & * & * & * \\ & & * & * & * \\ & 0 & & & * \end{bmatrix}$

$\implies$ span of 1st two columns, and span of 1st four columns are preserved.

$G/P = \{V_2 \subset V_4 \subset \mathbb{C}^5 : \mathbf{dim} V_2 = 2, \mathbf{dim} V_4 = 4\}$

G/P is a **partial flag variety**.

Projection Map:

$$B = \begin{bmatrix} * & * & \cdots & * & * \\ & * & \cdots & * & * \\ & & \ddots & \vdots & \vdots \\ & 0 & & * & * \\ & & & & * \end{bmatrix} \quad \subset \quad P = \begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & 0 & & * & * \\ & & & & * \end{bmatrix}$$

There is a natural map, $\pi : G/B \longrightarrow G/P$ via $gB \longmapsto gP$. Intuitively we forget some of the flag.

Example: $G = SL_5(\mathbb{C})$ and $P = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & & * & * & * \\ & & * & * & * \\ & 0 & & & * \end{bmatrix}$

$$\pi : V_1 \subset V_2 \subset V_3 \subset V_4 \subset \mathbb{C}^5 \longmapsto V_2 \subset V_4 \subset \mathbb{C}^5$$

Permutations:
The elements of W are words in the generating set $\{s_1, s_2, ..., s_k\}$

Example: $W = S_3$

$$w = 231 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = s_1 s_2$$

The **length** of w is $l(w) = 2$.

W has a partial order given by $v \leq u$ if and only if $\overline{BvB} \subset \overline{BuB}$ or equivalently, if some reduced word of v is a subword of u.

Example:
$u = s_1 s_2 s_3 s_4$ happens to be reduced. $v = s_2 s_4$ is a subword of $s_1 s_2 s_3 s_4$, therefore $v \leq u$.

Quotients on Permutations:

Choose $J \subset \{s_1, ..., s_{n-1}\}$, and let $W_P = \langle s_i | s_i \in J \rangle$. The quotient W/W_P can be described using the one-line notation of a permutation.

Example:

$G = SL_5(\mathbb{C}), \quad W = \langle s_1, s_2, s_3, s_4 \rangle, \quad W_P = \langle s_1, s_3 \rangle, \quad w = 23415$

Elements of W/W_P are determined by placing bars at positions 2 and 4 in one-line notation, because s_1 and s_2 were excluded from W_P.

$w = 23415 \quad \longrightarrow \quad 23|41|5$

$wW_P = \{23|14|5,\ 23|41|5,\ 32|14|5,\ 32|41|5\}$ (full coset)

We can always choose a minimal length representative $w^P = 23|14|5$.

Parabolic Decomposition: $w = w^P \cdot w_P = (23|14|5) \cdot (12|43|5)$

Schubert Varieties:
Given $u \in W$, define the Schubert variety $X_u = \overline{BuB} = \bigsqcup_{w \leq u} BwB$.

Let w_0 be the unique maximal length element of W. The opposite Borel $B^- = w_0 B w_0$.

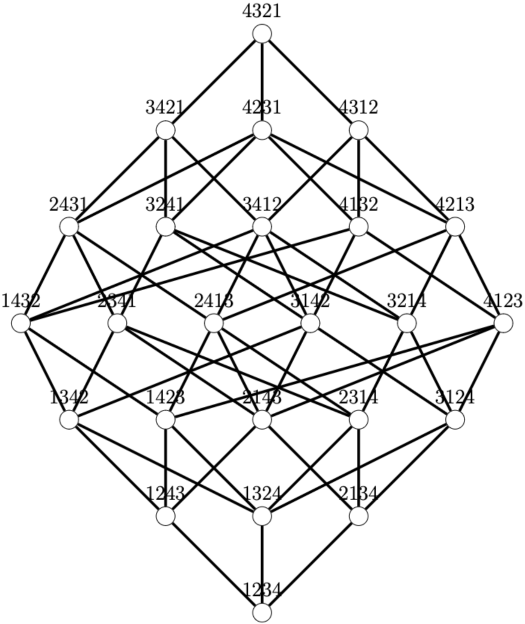
Given $v \in W$, define the opposite Schubert variety $X^v = \overline{B^- vB} = \bigsqcup_{v \leq w} B^- wB$.

Richardson Varieties:
The Richardson variety $\overline{R(v,u)} = X_u \cap X^v = X_u \cap w_0 X_{w_0 v}$.

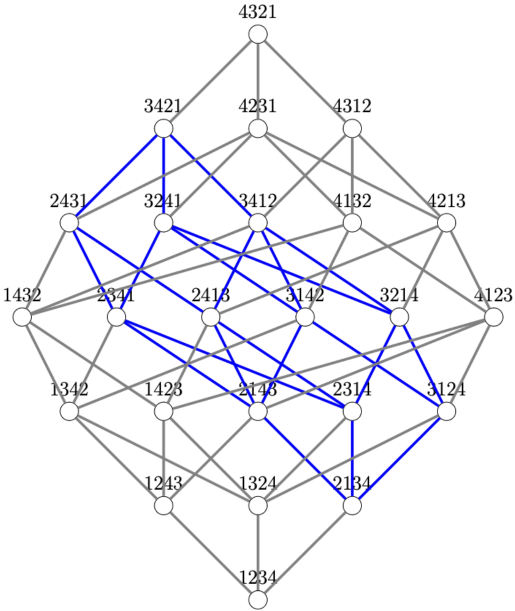
$\overline{R(v,u)}$ is nonempty if and only if $v \leq u$, and if it is nonempty then

$$\textbf{dim } \overline{R(v,u)} = l(u) - l(v).$$

Graded Bruhat Poset: $W = S_4$



Graded Bruhat Poset: $\overline{R(2134, 3421)} \subset G/B$



Historical Context:

- In 2012 Knutson-Lam-Speyer classified projected Richardson varieties using P-Bruhat order.
  - Projected Richardson varieties are normal and Cohen-Macaulay, and have rational resolutions
  - $\pi\left(\overline{R(v,u)}\right) = \pi\left(\overline{R(y,x)}\right)$ if and only if $[v,u]$ $[y,x]$ with respect to the P-Bruhat order.
- In 2017 Richmond-Slofstra studied $\pi : G/B \longrightarrow G/P$ in the context of describing smooth and rationally smooth Schubert varieties.
  - For any finite Lie Type, a Schubert variety is smooth if and only if it is an iterated fibre bundle of Grassmannians.
  - They showed $\pi : X_u \longrightarrow G/P$ has equidimensional fibers if and only if $u = u^P u_P$ is a BP-decomposition.
    (**I generalize this result to Richardson varieties**.)

Historical Context:

- In 2012 Billey-Coskun studied the singularities of projected Richardson varieties in Lie Type A. They showed that the singular locus of $\pi(R_v^u)$ denoted $\pi(R_v^u)^{sing}$, is the union of $(R_v^u)^{sing}$ with the points $gP \in \pi(R_v^u)$ with positive dimensional fibers.

- In 2022 Buch-Chaput-Mihalcea-Perrin studied fibers of projected Richardson varieties with an aim towards applications in quantum K-Theory. They defined a relaxation of transverse intersections called semitransverse intersections, and showed the generic fibers of projected Richardson varieties are a semitransverse intersection of a pair of Schubert varieties.

Transition to Combinatorics:
The geometry of restricting

$$\pi : G/B \longrightarrow G/P$$

to a Richardson variety is captured by the graded Bruhat poset and W/W_P.

Geometric Question:
Given $\pi : \overline{R(b,a)} \longrightarrow G/P$, when are the fibers equidimensional?

Geometric Answer:

Theorem (G.)

Let $\pi : \overline{R(b,a)} \longrightarrow G/P$ be the projection map to a partial flag variety, and let k equal the dimension of a generic fiber of π. The following are equivalent.

1. The fibers of π are equidimensional.
2. For each $\overline{R(v,u)} \subset \overline{R(b,a)}$ the generic fiber of $\pi|_{\overline{R(v,u)}}$ has dimension at most k.

Geometric Question:
Given $\pi : \overline{R(b,a)} \longrightarrow G/P$, when are the fibers equidimensional?

Combinatorial Answer:

Theorem (G.)

Let $\pi : \overline{R(b,a)} \longrightarrow G/P$. Let w_0 be the longest element of the Weyl group W associated to G/B. For $x \in W$ let $x^P x_P$ denotes the parabolic decomposition of x with respect to W/W_P, and $x \star y$ denote the Demazure product of x with y.Let $F(b,a)$ equal the dimension of a generic fiber of π.

$$F(b,a) = l(a_P) + l(w_{0P} b_P) - l\big(a_p \star (w_{0P} b_P)^{-1}\big)$$

$F(v,u)$ calculates the generic fiber dimension of the restriction $\pi\big|_{\overline{R(v,u)}}$.

Geometric Question:
Given $\pi : \overline{R(b,a)} \longrightarrow G/P$, when are the fibers equidimensional?

Combinatorial Answer:

Theorem (G.)

Let $\pi : \overline{R(b,a)} \longrightarrow G/P$ be the projection map to a partial flag variety, and let k equal the dimension of a generic fiber of π. The following are equivalent.

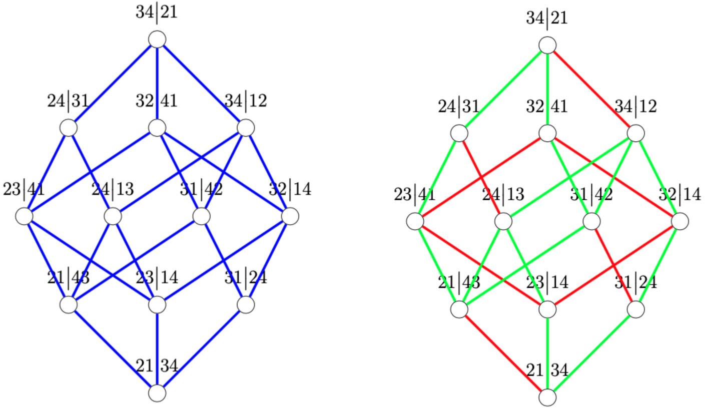
1. The fibers of π are equidimensional.
2. $F(v,u) \leq F(b,a)$ for each $[v,u] \subset [b,a]$.

Saturated Chains:

Color the graded Bruhat poset relative to W/W_P.
Elements in the same coset share a red edge , the other edges are green.

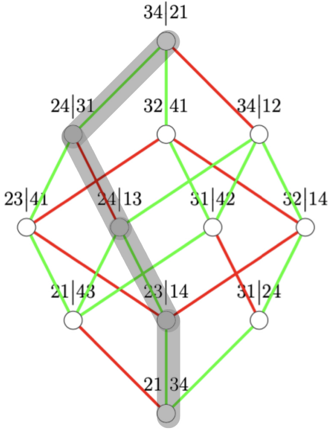
Example: $G = SL_4(\mathbb{C})$, $W_P = \langle s_1, s_3 \rangle$, and $\pi : \overline{R(2134, 3421)} \longrightarrow G/P$

Bruhat Interval $[2134, 3421]$ (cosets in red)



Saturated Chains:

Consider saturated chains on the colored Bruhat poset. The **weight** of a saturated chain is the number of red edges it contains.



A chain with weight 1.

Proposition (G.)

$F(v,u)$ equals the minimum weight of a saturated chain in $[v,u]$.

Proposition (G.)

Let $\mathcal{C}$ be a minimal weight chain in $[v,u] \subset [b,a]$. There exists a relative coset $xW_P \cap [b,a]$ with minimal weight chain $\mathcal{D}$ satisfying

$$\textbf{weight } \mathcal{C} \leq \textbf{weight } \mathcal{D}$$

When does $\pi : \overline{R(b,a)} \longrightarrow G/P$ have equidimensional fibers?

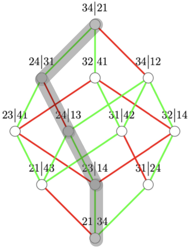
Summary:

- The generic fibers of π restricted to subvarieties each have smaller dimension than the generic fiber of π.

$$F(v,u) \leq F(b,a) \text{ for all } [v,u] \subset [b,a]$$

- Generic fibers of $\pi|_{\overline{R(v,u)}}$ have dimension equal to the minimal weight chain on $[v,u]$.

The weight of a chain comes from "coset steps".



- **Focus on Cosets:** Minimal weight chains in coset subintervals $[v,u]$ should take fewer "coset steps" than minimal weight chains in $[b,a]$.

Example 1:

$G = \mathsf{SL}_4(\mathbb{C}) \qquad W = S_4 = \langle s_1, s_2, s_3 \rangle$

$$G/B = \{V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4 : \mathbf{dim} V_k = k\}$$

Parabolic Subgroup:
Let $W_2 = \langle s_1, s_3 \rangle$ then $P_2 = BW_2B$ is a parabolic subgroup.

$$G/P_2 = \mathbf{Gr}_2(\mathbb{C}^4) = \{V \subset \mathbb{C}^4 : \mathbf{dim} V = 2\}$$

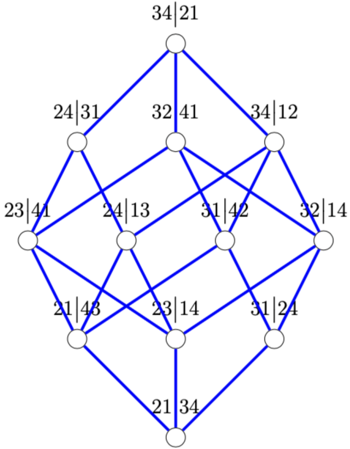
Projection Map:
The projection map $\pi : G/B \longrightarrow \mathbf{Gr}_2(\mathbb{C}^4)$ is defined by

$$V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4 \qquad \longmapsto \qquad V_2$$

Example 1:

Are the fibers of $\pi : \overline{R(2134, 3421)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^4)$ **equidimensional?**

The Combinatorial Picture:
Combinatorially $\overline{R(2134, 3421)}$ is represented by the Bruhat Interval $[2134, 3421]$.



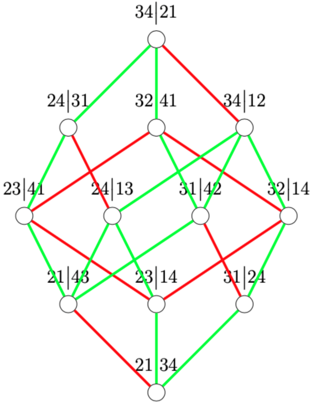
Example 1:

Are the fibers of $\pi : \overline{R(2134, 3421)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^4)$ **equidimensional?**

The Combinatorial Picture:

For projection we look at the cosets of W/W_2 in the interval.

Bruhat Interval $[2134, 3421]$ (cosets in red)



Example 1:

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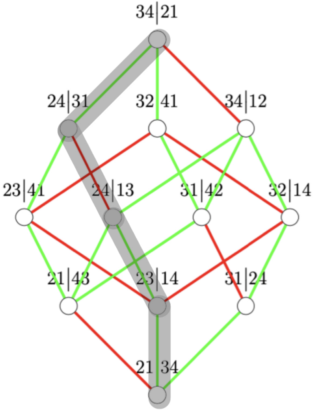


The least number of "coset steps" a saturated chain takes is one.

The generic fibers of $\pi : \overline{R(2134, 3421)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^4)$ are 1-dimensional.

Example 1:

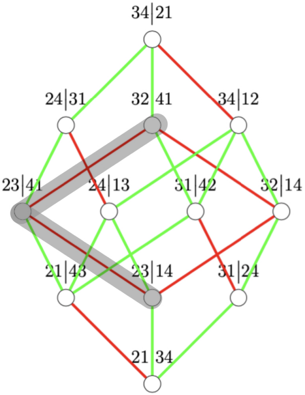
Are the fibers of $\pi : \overline{R(2134, 3421)} \longrightarrow$ Gr$_2(\mathbb{C}^4)$ equidimensional?

The Combinatorial Picture:

The saturated chains in the subinterval $[2314, 3241]$ must take two "coset" steps.

The generic fibers of the restriction $\pi : \overline{R(2314, 3241)} \longrightarrow$ **Gr**$_2(\mathbb{C}^4)$ are 2-dimensional.

The T-fixed point uB with $u = 2314$ is generic for the restriction and **Fiber**(uP) is $\mathbb{P} \times \mathbb{P}$, which is 2-dimensional.



THE FIBERS ARE NOT EQUIDIMENSIONAL.

Example 2:

$G = \mathsf{SL}_5(\mathbb{C}) \qquad W = S_5 = \langle s_1, s_2, s_3, s_4 \rangle$

$$G/B = \{V_1 \subset V_2 \subset V_3 \subset V_4 \subset \mathbb{C}^5 : \ \textbf{dim} V_k = k\}$$

Parabolic Subgroup:
Let $W_2 = \langle s_1, s_3, s_4 \rangle$ then $P_2 = BW_2B$ is a parabolic subgroup.

$$G/P_2 = \textbf{Gr}_2(\mathbb{C}^5) = \{V \subset \mathbb{C}^5 : \textbf{dim} V = 2\}$$

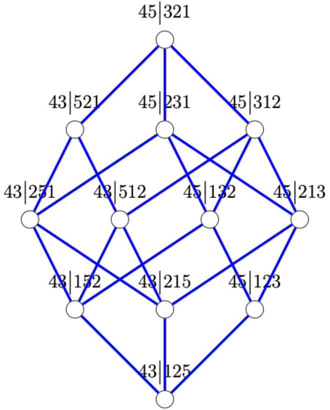
Projection Map:
The projection map $\pi : G/B \longrightarrow \textbf{Gr}_2(\mathbb{C}^5)$ is defined by

$$V_1 \subset V_2 \subset V_3 \subset V_4 \subset \mathbb{C}^5 \qquad \longmapsto \qquad V_2$$

Example 2:

Are the fibers of $\pi : \overline{R(43125, 45321)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^5)$ **equidimensional?**

The Combinatorial Picture:
Combinatorially $\overline{R(43125, 45321)}$ is represented by the Bruhat Interval $[43125, 45321]$.



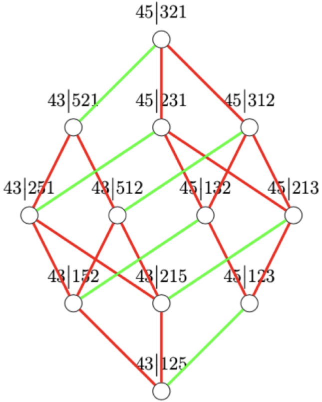
Example 2:

Are the fibers of $\pi : \overline{R(43125, 45321)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^5)$ equidimensional?

The Combinatorial Picture:

For projection we look at the cosets of W/W_2 in the interval.

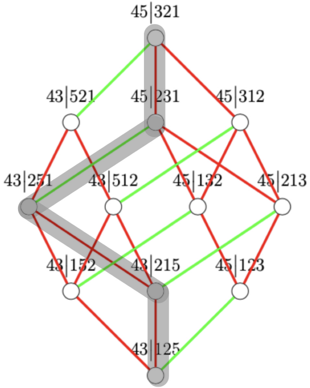
Bruhat Interval $[43125, 45321]$ (cosets in red)



Example 2:

Are the fibers of $\pi : \overline{R(43125, 45321)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^5)$ **equidimensional?**

The Combinatorial Picture:



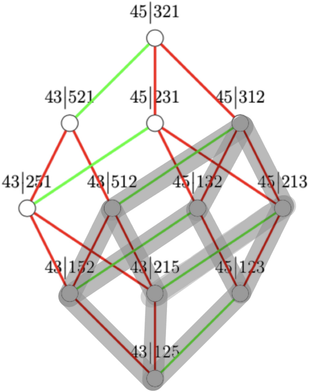
The least number of "coset steps" a saturated chain takes is 3.

The generic fibers of $\pi : \overline{R(43125, 45321)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^5)$ are 3-dimensional.

Example 2:

Are the fibers of $\pi : \overline{R(43125, 45321)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^5)$ **equidimensional?**

The Combinatorial Picture:



The saturated chains in subintervals take at most 3 coset steps.

Therefore each restriction's generic fiber has dimension less than the dimension of the generic fiber of $\pi : \overline{R(43125, 45321)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^5)$.

The minimal weight of saturated chains in subintervals is bounded by that of the parent interval $\longrightarrow$ π **has equdimensional fibers!**

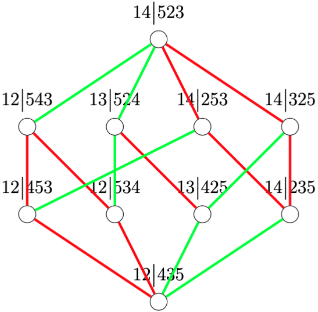
Theorem (G.)

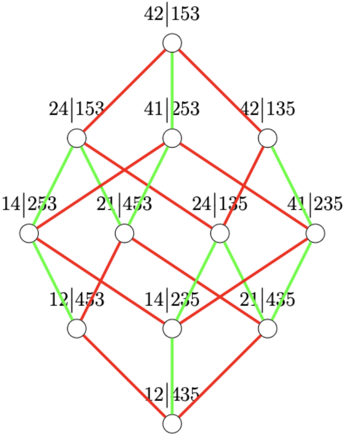
For each $x \in [b, a]$ define $w_P(x) = x_{\text{min}}^{-1} x_{\text{max}}$ where x_{min} and x_{max} are the unique minimum and maximum elements of $xW_P \cap [b, a]$. The following are equivalent.

1. The projection map $\pi : \overline{R(b, a)} \longrightarrow G/P$ has equidimensional fibers.
2. $F(v, u) \leq F(b, a)$ for each $[v, u] \subset [b, a]$.
3. Define $[b, a]_{\text{min}} = \{u_{\text{min}} : u \in [b, a]\}$. The interval $[b, a]$ has the set-theoretic factorization:

$$[b, a] = [b, a]_{\text{min}} \times [e, w_P(a)]$$

What Richardson vareity is being projected? Are the fibers equidimensional?





Proposition (G.)

Let $\pi : \overline{R(s_k, a)} \longrightarrow G/P$, where $G/P = Gr_r(\mathbb{C})$ and $s_k \in W_P = \langle S - \{s_r\} \rangle$. Let $w_P(x) = x_{\text{min}}^{-1} x_{\text{max}}$. Then π has equidimensional fibers if and only if:

(1) $w_P(a) = w_P(s_k)$,

(2) $l(w_P(a)) = F(s_k, a)$, and

(3)
  (i) whenever $k \leq r$, then $s_k s_{k+1} ... s_{r-1} s_r s_k \nleq a^P$
  (ii) whenever $k \geq r + 1$, then $s_k s_{k-1} ... s_{r+1} s_r s_k \nleq a^P$

Example:

Let $G = SL_4(\mathbb{C})$. Are the fibers of $\pi : \overline{R(2134, 3421)} \longrightarrow \mathbf{Gr}_2(\mathbb{C}^5)$ equidimensional?

$w_P(a) = a_{\text{min}}^{-1} a_{\text{max}} = (3412)(3421) = s_3$
$w_P(b) = b_{\text{min}}^{-1} b_{\text{max}} = (2134)(2143) = s_3$

$F(b, a) = 1$

$a^P = 34|12 = s_2 s_3 s_1 s_2$, and $s_k = 2134 = s_1 \to s_k ... s_r = s_1 s_2$

Since $s_1 s_2 \leq a^P = s_2 s_3 s_1 s_2$, then π **does not** have equidimensional fibers.

References

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Thank You