Explicit Modularity Results Via Hypergeometric Methods

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Classical Hypergeometric Functions

- Suppose that $r_i, q_i \in \mathbb{Q}$ for $1 \leq i \leq n$ with $q_1 = 1$. The generalized ${}_nF_{n-1}$ hypergeometric function is defined as

$$ {}_nF_{n-1}\left[\begin{matrix} r_1 & r_2 & \cdots & r_n \\ & q_2 & \cdots & q_n \end{matrix} ; \lambda\right] := \sum_{k=0}^{\infty} \frac{(r_1)_k (r_2)_k \cdots (r_n)_k}{(1)_k (q_2)_k \cdots (q_n)_k} \lambda^k $$

when $|\lambda| < 1$ and $n \geq 2$, where $(a)_k := a(a+1)\cdots(a+k-1)$.

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Example - The Fibonacci Numbers:

$$ F_n = \frac{n}{2^{n-1}} \cdot {}_2F_1\left[\begin{matrix} \frac{1-n}{2} & 1 - \frac{n}{2} \\ & \frac{3}{2} \end{matrix} ; 5\right] $$

for $n \geq 1$.

A Finite Field Hypergeometric Function

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- Now replace the Γ functions with Gauss sums $\mathfrak{g}$, reindex the sum to $\mathbb{F}_p^\times$, and normalize.

- Let $\lambda \in \mathbb{Q}$ and $\widehat{\mathbb{F}_p^\times} = \langle \omega \rangle$. In 2012, McCarthy defined

$$H_p \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ & q_2 & \cdots & q_n \end{bmatrix} ; \lambda \Bigg]$$

$$= \frac{1}{1-p} \sum_{k=0}^{p-2} \prod_{j=1}^{n} \frac{\mathfrak{g}(\omega^{k+(p-1)r_j})\mathfrak{g}(\overline{\omega}^{k+(p-1)q_j})}{\mathfrak{g}(\omega^{(p-1)r_j})\mathfrak{g}(\overline{\omega}^{(p-1)q_j})} \omega^k((-1)^n\lambda)$$

when $lcd(\alpha, \beta, \lambda) \mid (p-1)$ and $(p-1)r_j, (p-1)q_j \in \mathbb{Z}$ for all j.

Modular ℓ-adic Representations

- Important Remark: All representations we consider are finite-dimensional and semisimple so showing the traces agree is sufficient to connect Fourier coefficients $a_p(f)$ and H_p values on an arithmetic progression of primes.

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- Work of Deligne, Eichler, and Shimura gives a compatible family of 2-dimensional ℓ-adic representations ρ_f of $G_{\mathbb{Q}}$ attached to a cuspidal Hecke eigenform $f(\tau)$ of weight $k \geq 2$ and level N with

$$\mathrm{Tr}\rho_f(\mathrm{Frob}_p) = a_p(f),$$

where Frob_p denotes the Frobenius conjugacy class of G at $p \nmid \ell N$.

Hypergeometric ℓ-adic Representations

- Consider the datum $\mathrm{HD} = \{\{r_1, \ldots, r_n\}, \{1, q_2, \ldots, q_n\}\}$ with $\lambda \in \mathbb{Q}$, $M = \mathrm{lcd}(\{r_i\} \cup \{q_i\})$, ζ_M a primitive M-th root of unity, and $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

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- Work of Katz gives a compatible family of ℓ-adic representations $\rho_{\{\text{HD};\lambda\}}$ of $G_{\mathbb{Q}(\zeta_M)}$, where M is the least common denominator of the r_i and q_i.

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- In certain cases, the representation $\rho_{\{\text{HD};\lambda\}}$ can be extended to $G_{\mathbb{Q}}$. The dimension of $\rho_{\{\text{HD};\lambda\}}$ is n if $\lambda \neq 1$ and $n-1$ if $\lambda = 1$.

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- The trace is $\mathrm{Tr}\rho_{\{\mathrm{HD};\lambda\}}" = " H_p(\{r_i\}, \{q_i\}; \lambda)$, where "=" means the equality holds up to a power of p times a character value.

Extendability Explicitly

- $\mathrm{HD}_1 = \left\{\left\{\frac{1}{4}, \frac{1}{3}, \frac{2}{3}\right\}, \{1,1,1\}\right\}$ at $\lambda = 1$ <u>is extendable</u> from $G_{\mathbb{Q}(\zeta_{12})}$ to $G_{\mathbb{Q}}$, as

$$H_p\left[\begin{matrix} \frac{1}{4} & \frac{1}{3} & \frac{2}{3} \\ & 1 & 1 \end{matrix} ; 1\right] = H_p\left[\begin{matrix} \frac{3}{4} & \frac{1}{3} & \frac{2}{3} \\ & 1 & 1 \end{matrix} ; 1\right] \in \mathbb{Z}$$

for primes $p \equiv 1 \pmod{12}$.

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- $\mathrm{HD}_2 = \{\{\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\}, \{1,1,1\}\}$ at $\lambda = 1$ is **not** extendable from $G_{\mathbb{Q}(\zeta_6)}$ to $G_{\mathbb{Q}}$, as

$$H_p\left[\begin{matrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix} ; 1\right] = \phi(-1) H_p\left[\begin{matrix} \frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix} ; 1\right] \in \mathbb{Z}[\sqrt{-3}]$$

for primes $p \equiv 1 \pmod{6}$.

Known Modularity for $n = 3$ and $\lambda = 1$

Mortenson, 2004

$$H_p\left[\begin{matrix} \frac{1}{2} & \frac{1}{d} & 1-\frac{1}{d} \\ & 1 & 1 \end{matrix} ; 1\right] = a_p(f_d(\tau))$$

for $d \in \{2,3,4,6\}$ and primes $p \equiv 1 \pmod{d}$, where $f_2(\tau) = f_6(\tau) = \eta(4\tau)^6$, $f_3(\tau) = \eta(2\tau)^3\eta(6\tau)^3$, and $f_4(\tau) = \eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$.

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McCarthy and Papanikolas, 2015

$$H_p\left[\begin{matrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1\end{matrix}\ ;\ 1\right] = a_p(f_{32.3.c.a})$$

for primes $p \equiv 1 \pmod{4}$.

The Conjectures of Dawsey and McCarthy

- In 2020, Dawsey and McCarthy gave exact formulas for order four complete subgraphs of Generalized Paley graphs in terms of finite field ${}_3F_2(1)$ values. They found two new conjectural relations, similar to those on the last slide.

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- A year later, the same authors found 13 more relations numerically. In total, the 15 new conjectures have the form

$$H_p\left[\begin{matrix} r_1 & r_2 & 1-r_2 \\ & 1 & 1 \end{matrix} ; 1\right] = \chi_{(r_1,r_2)}(p) \cdot a_p(f_{\{r_1,r_2\}})$$

for $r_1, r_2 \in \mathbb{Q} \cap (0,1)$ and primes $p \equiv 1 \pmod{M}$, where $M = \operatorname{lcd}(r_1, r_2)$, $\chi_{(r_1,r_2)}$ is a character of order at most two, and $f_{\{r_1,r_2\}}(\tau)$ is the associated weight three newform.

Our Results

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Theorem (Allen, G., Long, Tu)

The conjectures of Dawsey and McCarthy are true in the following six cases

$$(r_1, r_2) \in \left\{ \left(\frac{1}{8}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{4}, \frac{1}{3}\right), \left(\frac{1}{6}, \frac{1}{3}\right), \left(\frac{1}{2}, \frac{1}{10}\right), \left(\frac{1}{2}, \frac{1}{12}\right) \right\}.$$

The modular forms are

$$f_{\{\frac{1}{8},\frac{1}{2}\}}(\tau) = f_{64.3.d.a}, \ f_{\{\frac{1}{3},\frac{1}{3}\}}(\tau) = f_{27.3.b.b}, \ f_{\{\frac{1}{4},\frac{1}{3}\}}(\tau) = f_{36.3.d.a}$$

$$f_{\{\frac{1}{6},\frac{1}{3}\}}(\tau) = f_{108.3.c.b}, \ f_{\{\frac{1}{2},\frac{1}{10}\}}(\tau) = f_{400.3.b.b}, \ f_{\{\frac{1}{2},\frac{1}{12}\}}(\tau) = f_{24.3.h.a}.$$

Explicitly Calculating the Modular Form - I

The Euler integral formula for the corresponding classical ${}_3F_2$ with $(r_1, r_2) = \left(\frac{1}{8}, \frac{1}{2}\right)$ is

$${}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{8} \\ & 1 & 1\end{matrix}\,;\ 1\right] = B\left(\frac{1}{8}, \frac{7}{8}\right)^{-1} \int_0^1 t^{-\frac{7}{8}}(1-t)^{-\frac{1}{8}} \cdot {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}\,;\ t\right] dt$$

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Now

$$ \left(t^{-\frac{7}{8}}(1-t)^{-\frac{1}{8}} \cdot {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix} ;\ t\right] \frac{dt}{d\tau}\right)_{t=\lambda(\tau)} $$

$$ = \left(\sqrt{2}\pi i\right) \eta(\tau)^6 \left(\frac{\eta\left(\frac{\tau}{2}\right)}{\eta(2\tau)}\right)^3 $$

Explicitly Calculating the Modular Form - II

- Recall the modular form for the case $(r_1, r_2) \in \left(\frac{1}{8}, \frac{1}{2}\right)$ is $f_{\{\frac{1}{8}, \frac{1}{2}\}}(\tau) = f_{64.3.d.a}$.

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- If $p \equiv 1 \pmod 8$ we find that

$$a_p\left(\eta(16\tau)^6\left(\frac{\eta(8\tau)}{\eta(32\tau)}\right)^3\right) = a_p(f_{256.3.c.g}).$$

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- If $p \equiv 1 \pmod 8$ then $a_p(f_{64.3.d.a}) = \chi_8(-1)a_p(f_{256.3.c.g})$.

The Hypergeometric Modularity Method - I

- Key Observation: It suffices to show

$$H_p\left[\begin{matrix} r_1 & r_2 & 1-r_2 \\ & 1 & 1 \end{matrix} ; 1\right] \equiv a_p(f_{\{r_1,r_2\}}) \pmod{p^2}$$

for primes $p \equiv 1 \pmod{M}$.

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- Let $F_s(r_1, r_2)$ denote the truncation of the classical ${}_3F_2(1)$ with first row $\{r_1, r_2, 1-r_2\}$ and second row $\{1,1,1\}$ at $p^s - 1$.

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- Commutative Formal Group Laws: Comparing the q and t expansions for the classical ${}_3F_2(1)$ gives a congruence of the form

$$F_1(r_1, r_2) \equiv \psi_{(r_1,r_2)}(p) \cdot a_p(f_{\{r_1,r_2\}}) \pmod{p}$$

for $p \equiv 1 \pmod{M}$.

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- The Gross-Koblitz Formula and a Supercongruence: Connects the H_p function and truncated ${}_3F_2(1)$

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- A Dwork p-adic Unit Root Supercongruence:

$$F_{s+1}(r_1, r_2) \equiv F_s(r_1, r_2) F_1(r_1, r_2) \pmod{p^2}$$

for primes $p \equiv 1 \pmod{M}$.

Thanks for Your Attention!