The low point in the theta cycle of modular forms modulo p^2

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Contents

Basic Definitions

Throughout, $p \geq 5$ is a prime, and $M_k := M_k(\mathbb{Z}_{(p)})$

Modular forms modulo p^m

$$M_k(\mathbb{Z}/p^m\mathbb{Z}) := M_k(\mathbb{Z}) \otimes \mathbb{Z}/p^m\mathbb{Z}$$

if $f \in M_k$, write $\overline{f} \in M_k(\mathbb{Z}/p^m\mathbb{Z})$, and if $f, g \in M_k$ with $\overline{f} = \overline{g}$ in $M_k(\mathbb{Z}/p^m\mathbb{Z})$, write $f \equiv g \pmod{p^m}$.

Basic Definitions

Definition

For $f \in M_k$, the **mod** p^m **filtration** of f is

$$w_{p^m}(f) := \inf\{k' : \overline{f} = \overline{g} \text{ for } g \in M_{k'}\}.$$

Example

- $E_2 \equiv E_{p+1} \pmod{p}$
- $E_{p-1} \equiv 1 \pmod{p}, \quad E_{p-1}^p \equiv 1 \pmod{p^2}$
- $E_{p+3} \equiv E_4 \pmod{p}$

Basic Definitions

Ramanujan theta operator $\theta := q\frac{d}{dq}$ on q-series:

$$\theta\left(\sum a(n)q^n\right) = \sum na(n)q^n.$$

Fact: If $f \in M_k$ then $\overline{\theta f} \in M_{k'}(\mathbb{Z}/p^m\mathbb{Z})$ (here, $k' = w_{p^m}(f) + 2 + 2\varphi(p^m)$).

Definition

The **theta cycle mod** p^m of $f \in M_k$ is

$$\Omega_{p^m}(f) := \left(w_{p^m}(\theta^m f), w_{p^m}(\theta^{m+1} f), \ldots, w_{p^m}(\theta^{\varphi(p^m)+m-1} f)\right)$$

For technical reasons...

$$\tilde{\Omega}_{p^m}(f) := \left(w_{p^m}(f), w_{p^m}(\theta f), \ldots, w_{p^m}(\theta^{\varphi(p^m)+m-1} f)\right)$$

Basic Definitions

Describing $\Omega_{p^m}(f)$ can give important information about f. In particular, we want to know...

1. How does $\Omega_{p^m}(f)$ increase?
2. Where are its high and low points?

Example

- $\Omega_p(E_{p-1})$ is trivial.
- $\tilde{\Omega}_p(E_{p+1}) = (p+1, 2p+2, \ldots)$ can be computed directly:

$$\theta(E_{p+1}) \equiv \frac{1}{12}\left(E_{p+1}^2 - E_{p-1}E_{p+3}\right) \pmod{p}.$$

Motivation

$\boxed{m = 1}$ $\Omega_p(f)$ is well-understood.

- Tate, Jochnowitz: Position/filtration of high/low points known (combinatorial argument).
  - Always one or two low points
  - If $k < p$, position of 1st low point determines whether f is ordinary at p or not, i.e. has U_p-congruence or not:

$$a(np) \equiv 0 \pmod{p} \quad \forall n$$

- Ahlgren-Boylan: Classification of Ramanujan congruences for $p(n)$.
  - ...and work extending these results for more general classes of forms (J. Sinick, H-Smith).

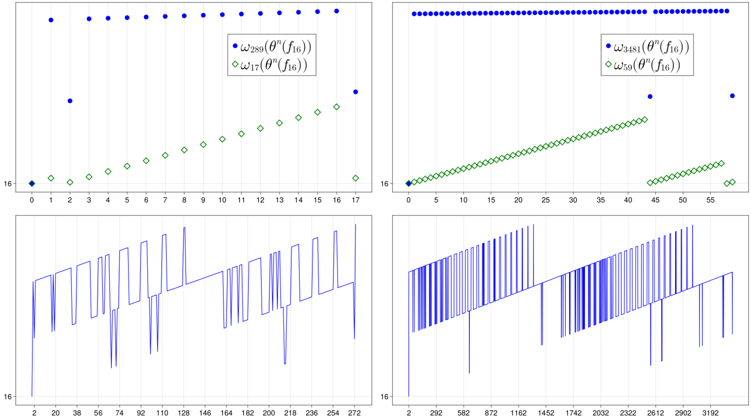
Motivation

$\boxed{m \geq 2}$ Very little known about $\Omega_{p^m}(f)$!

- Chen-Kiming: If $w_p(f) = k \not\equiv 0 \pmod{p}$ then
$$w_{p^2}(\theta f) = k + 2 + 2p(p-1).$$
- Kim-Lee: For $n_t = tp$ or $n_t = tp - k + 1$, under some assumptions,
$$w_{p^m}(\theta^{n_t} f) \leq k + 2n_t + p^{m-1}(p-1).$$
Also some exact results when $m = 2$ at each n_t.
- Knowing more about $\Omega_{p^m}(f)$ could potentially provide information about $f \pmod{p^m}$, e.g. if f has U_p-congruence mod p^m.

Strategy to compute $\Omega_{p^2}(f)$

We can't use counting arguments here... $\Omega_{p^2}(f)$ is very erratic in general.



Strategy to tackle $m = 2$

Direct approach: Let $f \in M_k$.

1. $\theta(f) = \frac{k}{12} E_2 f + g_{k+2}$ for some $g_{k+2} \in M_{k+2}$
2. Iterate θ in this way:

$$\theta^n(f) = \alpha_{n,k} E_2^n f + G$$

   for some form G.

3. Find "minimal" expression for $E_2 \pmod{p^2}$ and do more stuff to read $w_{p^2}(\theta^n f)$ from resulting expression.

E_2 congruence

Theorem (H, Raum, Richter)

For $p \geq 5$ prime,

$$E_2 \equiv E_{p-1}^{p-2} \left(E_{p-1}^{p+1} f_{p+1} + pE_{p+1}^{p} \right) \pmod{p^2}$$

for some $f_{p+1} \in M_{p+1}$.

E_2 congruence

Theorem (H, Raum, Richter)

For $p \geq 5$ prime,

$$E_2 \equiv E_{p-1}^{p-2}\left(E_{p-1}^{p+1} f_{p+1} + pE_{p+1}^{p}\right) \pmod{p^2}$$

for some $f_{p+1} \in M_{p+1}$.

Using this with $\theta^n(f) = \alpha_{n,k} E_2^n f + G$ isn't quite enough because we can't control G. However, $\Omega_p(f)$ gives some information about $\Omega_{p^2}(f)$ (divisibility by E_{p-1}).

Initial elements of $\Omega_{p^2}(f)$

Theorem (HRR)

Before the 1st low point j of $\tilde{\Omega}_p(f)$, $\tilde{\Omega}_{p^2}(f)$ increases by 2 each step, except for the first step which increases by $2 + 2p(p-1)$ (Chen-Kiming). Thus,

$$w_{p^2}(\theta^i f) = k + 2i + 2p(p-1), \quad 1 \leq i < j.$$

Initial elements of $\Omega_{p^2}(f)$

In conjunction with Kim-Lee...

Corollary

The position of the 1st low point of $\tilde{\Omega}_p(f)$ and of $\tilde{\Omega}_{p^2}(f)$ coincide. At this low point, we have the bounds:

- $k \not\equiv 1, 2 \pmod p$, $k \equiv k_0 \pmod p$:

$$w_{p^2}(\theta^{p+1-k_0} f) \leq k + 2 - 2k_0 + p(p+1)$$

- $k \equiv 1 \pmod p$:

$$w_{p^2}(\theta^p f) \leq k + p(p+1)$$

- $k \equiv 2 \pmod p$:

$$w_{p^2}(\theta^{p-1} f) \leq k - 2 + p(p+1).$$

Initial elements of $\Omega_{p^2}(f)$

Corollary

If $k \leq p+1$ and f is ordinary at p, then low points have exact filtrations

- $k < p$:
$$w_{p^2}(\theta^{p-k+1} f) = 2 - k + p(p+1)$$
- $k = p+1$:
$$w_{p^2}(\theta^{p} f) = (p+1)^2$$

Theta cycle of E_{p-1} mod p^2

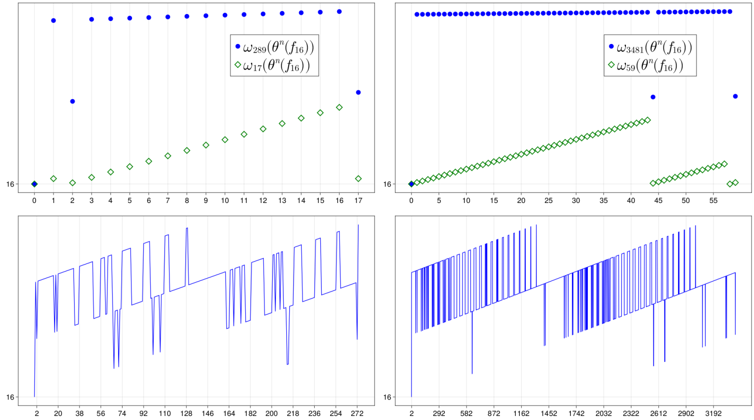
In contrast with the trivial $\Omega_p(E_{p-1})$, $\Omega_{p^2}(E_{p-1})$ is very regular.

Theorem (HRR)

There are exactly p low points in $\Omega_{p^2}(E_{p-1})$ with rises by $p+1$ in between. These low points occur at $\theta^{i(p-1)+2}(E_{p-1})$ for $0 \leq i < p$, and the filtrations are

$$w_{p^2}(\theta^{i(p-1)+2}E_{p-1}) = 2p+2.$$

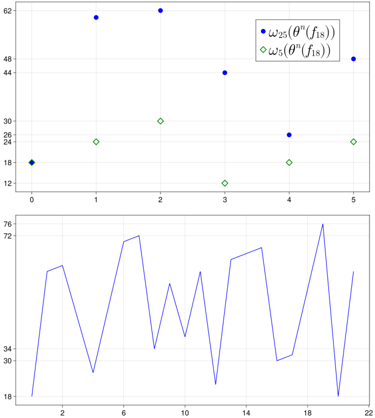
Some graphs



No U_{17}-congruence mod 17, No U_{17}-congruence mod 17^2

Has U_{59}-congruence mod 59, No U_{59}-congruence mod 59^2

Some graphs



f_{18}, f_{20} both have U_p-congruences $\pmod{p^2}$.

Thank you!!!