The low point in the theta cycle of modular forms modulo p^2

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Contents

Basic Definitions

Throughout, $p \geq 5$ is a prime, and $M_k := M_k(\mathbb{Z}_{(p)})$

Modular forms modulo p^m

$$M_k(\mathbb{Z}/p^m\mathbb{Z}) := M_k(\mathbb{Z}) \otimes \mathbb{Z}/p^m\mathbb{Z}$$

if $f \in M_k$, write $\overline{f} \in M_k(\mathbb{Z}/p^m\mathbb{Z})$, and if $f, g \in M_k$ with $\overline{f} = \overline{g}$ in $M_k(\mathbb{Z}/p^m\mathbb{Z})$, write $f \equiv g \pmod{p^m}$.

Basic Definitions

Definition

For $f \in M_k$, the mod p^m filtration w_{p^m} of f is

$$w_{p^m}(f) := \inf\{k' : \bar{f} = \bar{g} \text{ for } g \in M_{k'}\}.$$

Example

- $E_2 \equiv E_{p+1} \pmod p$
- $E_{p-1} \equiv 1 \pmod p$, $\quad E_{p-1}^{p} \equiv 1 \pmod{p^2}$
- $E_{p+3} \equiv E_4 \pmod p$

Basic Definitions

Ramanujan theta operator $\Theta := q\frac{d}{dq}$ on q-series:

$$\Theta \sum a(n)q^n = \sum n a(n) q^n.$$

Fact: If $f \in M_k$ then $\overline{\Theta f} \in M_{k'}(\mathbb{Z}/p^m\mathbb{Z})$ (here, $k' = w_{p^m}(f) + 2 + 2\varphi(p^m)$).

Definition

The theta cycle mod p^m of $f \in M_k$ is

$$\Theta_{p^m}(f) := \left(w_{p^m}(\Theta^m f), w_{p^m}(\Theta^{m+1} f), \ldots, w_{p^m}(\Theta^{\varphi(p^m)+m-1} f)\right)$$

For technical reasons...

$$\tilde{\Theta}_{p^m}(f) := \left(w_{p^m}(f), w_{p^m}(\Theta f), \ldots, w_{p^m}(\Theta^{\varphi(p^m)+m-1} f)\right)$$

Basic De nitions

Describing $_{p^m}(f)$ can give important information about f. In particular, we want to know...

1. How does $_{p^m}(f)$ increase?
2. Where are its high and low points?

Example

- $_p(E_{p \ 1})$ is trivial.
- $\tilde{}_p(E_{p+1}) = (p+1; 2p+2; : : :)$ can be computed directly:

$$(E_{p+1}) \quad \frac{1}{12} \ E_{p+1}^2 \quad E_{p \ 1} E_{p+3} \quad (\text{mod } p):$$

Motivation

m = 1 $\ _p(f)$ is well-understood.

- Tate, Jochnowitz: Position/ ltration of high/low points known (combinatorial argument).
  - Always one or two low points
  - If $k < p$, position of 1st low point determines whether f is ordinary at p or not, i.e. has U_p-congruence or not:

$$a(np) \quad 0 \pmod{p} \quad 8n$$

- Ahlgren-Boylan: Classi cation of Ramanujan congruences for $p(n)$.
  - ...and work extending these results for more general classes of forms (J. Sinick, H-Smith).

Motivation

m 2	Very little known about $_{p^m}(f)$!

- Chen-Kiming: If $w_p(f) = k$ 6 0 (mod p) then

$$w_{p^2}(\; f) = k + 2 + 2\,p(p \quad 1):$$

- Kim-Lee: For $n_t = tp$ or $n_t = tp \quad k + 1$, under some assumptions,

$$w_{p^m}(\; ^{n_t} f) \quad k + 2n_t + p^{m \quad 1}(p \quad 1):$$

  Also some exact results when $m = 2$ at each n_t.

- Knowing more about $_{p^m}(f)$ could potentially provide information about f (mod p^m), e.g. if f has U_p-congruence mod p^m.

Strategy to compute $_{p^2}(f)$

We can't use counting arguments here...$_{p^2}(f)$ is very erratic in general.

Strategy to tackle $m = 2$

Direct approach: Let f 2 M_k.

1. $(f) = \frac{k}{12} E_2 f + g_{k+2}$ for some g_{k+2} 2 M_{k+2}

2. Iterate in this way:

$$^{n}(f) = \quad _{n;k} E_2^n f + G$$

   for some form G.

3. Find \minimal" expression for E_2 (mod p^2) and do more stu to read $w_{p^2}(\, ^n f)$ from resulting expression.

E_2 congruence

Theorem (H, Raum, Richter)

For $p \geq 5$ prime,

$$E_2 \equiv E_{p-1}^{p-2} \; E_{p-1}^{p+1} f_{p+1} + pE_{p+1}^{p} \pmod{p^2}$$

for some $f_{p+1} \in M_{p+1}$.

E_2 congruence

Theorem (H, Raum, Richter)

For p   5 prime,

$$E_2 \quad E_{p\;\;1}^{p\;\;2} \; E_{p\;\;1}^{p+1} f_{p+1} + pE_{p+1}^{p} \qquad \pmod{p^2}$$

for some f_{p+1} 2 M_{p+1} .

Using this with $^n(f) = \;\; {}_{n;k} E_2^n f + G$ isn't quite enough because we can't control G. However, $_p(f)$ gives some information about $_{p^2}(f)$ (divisibility by $E_{p\;\;1}$).

Initial elements of $_{p^2}(f)$

Theorem (HRR)

Before the 1st low point j of $\tilde{}_p(f)$, $\tilde{}_{p^2}(f)$ increases by 2 each step, except for the rst step which increases by $2 + 2p(p \quad 1)$ (Chen-Kiming). Thus,

$$w_{p^2}(\, {}^i f) = k + 2i + 2p(p \quad 1); \quad 1 \quad i < j:$$

Initial elements of $_{p^2}(f)$

In conjunction with Kim-Lee...

Corollary

The position of the 1st low point of $\tilde{\ }_p(f)$ and of $\tilde{\ }_{p^2}(f)$ coincide.
At this low point, we have the bounds:

- $k\ 6\ 1;2 \pmod p$, $k \quad k_0 \pmod p$:

$$w_{p^2}(\ ^{p+1}\ \ ^{k_0}f) \quad k+2 \quad 2k_0 + p(p+1)$$

- $k \quad 1 \pmod p$:

$$w_{p^2}(\ ^{p}f) \quad k + p(p+1)$$

- $k \quad 2 \pmod p$:

$$w_{p^2}(\ ^{p}\ \ ^{1}f) \quad k \quad 2 + p(p+1):$$

Initial elements of $_{p^2}(f)$

Corollary

If $k \quad p+1$ and f is ordinary atp, then low points have exact Itrations

- $k < p$:

$$w_{p^2}(\ ^{p} \ ^{k+1} f) = 2 \quad k + p(p+1)$$

- $k = p+1$:

$$w_{p^2}(\ ^{p} f) = (p+1)^2$$

Theta cycle of E_{p-1} mod p^2

In contrast with the trivial $\Omega_p(E_{p-1})$, $\Omega_{p^2}(E_{p-1})$ is very regular.

Theorem (HRR)

There are exactly p low points in $\Omega_{p^2}(E_{p-1})$ with rises by $p+1$ in between. These low points occur at $\theta^{i(p-1)+2}(E_{p-1})$ for $0 \le i < p$, and the θ filtrations are

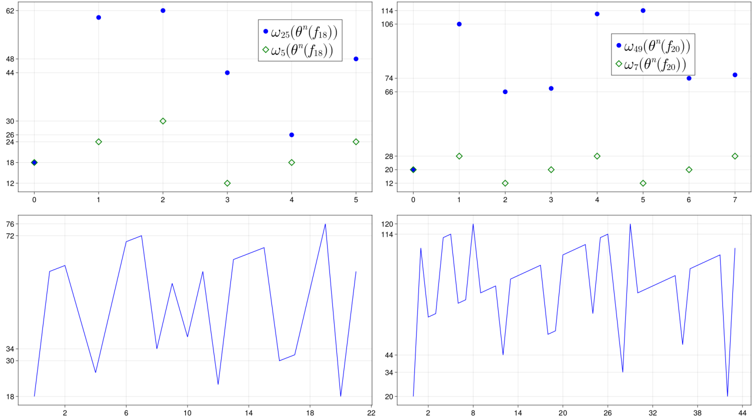
$$w_{p^2}(\theta^{i(p-1)+2}E_{p-1}) = 2p + 2.$$

Some graphs

No U_{17}-congruence mod 17,
Has U_{59}-congruence mod 59,

No U_{17}-congruence mod 17^2
No U_{59}-congruence mod 59^2

Some graphs



$f_{18}; f_{20}$ both have U_p-congruences $\pmod{p^2}$.