# Differential operators acting on Jacobi forms and Poincaré series

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# Introduction

- $M_k$ - vector space of modular forms of weight $k$ over $SL_2(\mathbb{Z})$.
- $S_k$ - subspace of cusp forms of weight $k$ over $SL_2(\mathbb{Z})$.
- $(f\,|_k\,\gamma)(\tau) := (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right), \tau \in \mathbb{H}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.
- $f \in M_k \implies f\,|_k\,\gamma = f \left( f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \right)$ for all $\gamma \in SL_2(\mathbb{Z})$.
- $f'\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2} f'(\tau) + \frac{k}{2\pi i}\, c(c\tau+d)^{k+1} f'(\tau)$.
- **Ramanujan-Serre derivative:** $\vartheta_k f := \frac{1}{2\pi i}\frac{df}{d\tau} - \frac{k}{12} E_2 f$.
- $\vartheta_k : M_k \longrightarrow M_{k+2}$ is linear.

# Rankin-Cohen brackets of modular forms

- **Rankin (1956):** Polynomials involving derivatives of modular forms.
- **Cohen (1975):** $f \in M_k, g \in M_l$ and $\nu \geq 0$

$$[f, g]_\nu := \sum_{r=0}^{\nu} (-1)^r \binom{k+\nu-1}{\nu-r} \binom{l+\nu-1}{r} f^{(r)} g^{(\nu-r)}.$$

- $[f, g]_\nu \in M_{k+l+2\nu}$.
- $[f, g]_\nu \in S_{k+l+2\nu}$ for $\nu > 0$.
- $[\cdot, \cdot]_\nu : M_k \times M_l \longrightarrow M_{k+l+2\nu}$ is bilinear.

# Examples

- $E_k(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k \gamma, \quad \Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{Z} \right\}.$
- $E_k \in M_k \ (k \geq 4).$
- $P_{k,m}(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i m \tau}|_k \gamma.$
- $P_{k,m} \in S_k \ (k \geq 12).$
- **Petersson Scalar product:** $f, g \in S_k$

$$\langle f, g \rangle := \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} f(\tau)\overline{g(\tau)} v^{k-2} du dv, \ \ \tau = u + iv.$$

- $\langle f, P_{k,m} \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m), \ f(\tau) = \sum_{n \geq 1} a_f(n) q^n \in S_k.$

# Special values of $L$-functions

$$f(\tau) = \sum_{n\geq 1} a_f(n) q^n \in S_*, \quad g(\tau) = \sum_{n\geq 1} a_g(n) q^n \in S_l$$

- **Rankin (1952):** $\langle f, gE_k\rangle = C \sum_{n\geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^{k+l-1}}$
- **Zagier (1976):** $\langle f, [E_k, g]_\nu\rangle = C \sum_{n\geq 1} \frac{a_f(n)\overline{a_g(n)}}{n^{k+l+2\nu-1}}$
- **Kohnen (1991):** $\langle f, gP_{k,m}\rangle = C \sum_{n\geq 1} \frac{a_f(n)\overline{a_g(n+m)}}{(n+m)^{k+l-1}}$
- **Herrero (2014):** $\langle f, [P_{k,m}, g]_\nu\rangle = C \sum_{n\geq 1} \frac{a_f(n)\overline{a_g(n+m)}[e^{2\pi i n\tau}, e^{2\pi i m\tau}]_\nu(0)}{(n+m)^{k+l+2\nu-1}}$

# Rankin-Cohen bracket and Poincaré series

- Assume that $f \in M_k$ and $g$ is a complex-valued holomorphic function defined on $\mathbb{H}$ such that $[f, g]_\nu \in M_{k+l+2\nu}$. What can we say about $g$?
- **Choie and Lee (2011):** $g$ satisfies the transformation properties of modular forms of weight $l$ over $SL_2(\mathbb{Z})$.

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- $\mathbb{P}_k(f) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f|_k \gamma, \quad f(q) = f(e^{2\pi i \tau}) = \sum_{n \geq 1} a_f(n) q^n.$

- $\mathbb{P}_k(1) = E_k, \ \ \mathbb{P}_k(e^{2\pi i m \tau}) = P_{k,m}.$

- **Williams (2018):** $l \in 2\mathbb{N}$ with $l \geq 4$ and $k, \nu, m \in \mathbb{N}, \ f \in M_k$ and $l \geq k+2$ (if $f$ is not a cusp form)

$$[f, P_{l,m}]_\nu = \mathbb{P}([f, e^{2\pi i m \tau}]_\nu).$$

# Jacobi forms

- $\Gamma^J := \{(M, X) : M \in SL_2(\mathbb{Z}),\ X = (\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}\}$
- **Group operation:** $(M, X)(M', X') = (MM', XM' + X')$.
- **Action of $\Gamma^J$ on $\mathbb{H} \times \mathbb{C}$ :**

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right).$$

- **Slash operator:** Fix $k, m \in \mathbb{Z}$. For $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ holomorphic and $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma^J$ define

$$(\phi|_{k,m}\gamma)(\tau, z) := (c\tau + d)^{-k} e\left(\frac{-c}{c\tau + d} m(z + \lambda\tau + \mu)^2 + m\lambda^2\tau + 2\lambda m z\right) \phi(\gamma \cdot (\tau, z))$$

# Jacobi forms

## Definition

A holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is said to be a Jacobi form of weight $k$ and index $m$ if it satisfies

$$\phi|_{k,m}\gamma = \phi \quad \text{for every } \gamma \in \Gamma^J$$

and $\phi$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z} \\ 4nm \geq r^2}} c_\phi(n,r) q^n \zeta^r.$$

- If $c_\phi(n,r) = 0$ for $4nm = r^2$, then $\phi$ is called a Jacobi cusp form.
- We denote the space of all Jacobi (resp. cusp) forms of weight $k$ and index $m$ by $J_{k,m}$ (resp. $J_{k,m}^{cusp}$).

# Jacobi forms: Examples

- **Jacobi Eisenstein series:**

$$E_{k,m}(\tau,z) := \sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} 1|_{k,m}\gamma,\quad \Gamma^J_\infty := \left\{\left(\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix},(0,\mu)\right) : \lambda,\mu\in\mathbb{Z}\right\}.$$

- $E_{k,m}\in J_{k,m}$ $(k\geq 4)$.
- **Jacobi Poincaré series:**

$$P^{(n,r)}_{k,m}(\tau,z) := \sum_{\gamma\in\Gamma^J_\infty\backslash\Gamma^J} e^{2\pi i(n\tau+rz)}|_{k,m}\gamma.$$

- $P_{k,m}\in J^{cusp}_{k,m}(k\geq 10)$.
- **Petersson Scalar product:** $\phi,\psi\in J^{cusp}_{k,m}$

$$\langle\phi,\psi\rangle := \int_{\Gamma^J\backslash\mathbb{H}\times\mathbb{C}} \phi(\tau,z)\overline{\psi(\tau,z)}v^{k-3}e^{\frac{-\pi my^2}{v}}dudvdxdy,\quad \tau=u+iv, z=x+iy.$$

- $\langle\phi,P^{(n,r)}_{k,m}\rangle = \dfrac{m^{k-2}\Gamma(k-3/2)}{2\pi^{k-3/2}(4mn-r^2)^{k-3/2}}c_\phi(n,r).$

# Rankin-Cohen bracket of Jacobi forms

- **Heat operator:** $L_m := \frac{1}{(2\pi i)^2}\left(8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}\right)$.
- $L_m \phi|_{k+2,m}\gamma = L_m\phi + \frac{(2k-1)mc}{\pi i(c\tau + d)}\phi$.
- **Modified Heat Operator:** $L_{k,m} := L_m - \frac{2k-1}{6}E_2 : J_{k,m} \longrightarrow J_{k+2,m}$ is linear.
- **Choie (1997):** $\phi, \psi : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$

$$[\phi, \psi]_\nu := \sum_{l=0}^{\nu} (-1)^l \binom{k_1 + \nu - \frac{3}{2}}{\nu - l} \binom{k_2 + \nu - \frac{3}{2}}{l} m_1^{\nu - l} m_2^l L_{m_1}^l(\phi) L_{m_2}^{\nu - l}(\psi).$$

- $[\phi|_{k_1,m_1}\gamma, \psi|_{k_2,m_2}\gamma]_\nu = [\phi, \psi]|_{k_1+k_2+2\nu, m_1+m_2}\gamma, \quad \forall \gamma \in \Gamma^J$.
- If $\phi \in J_{k_1,m_1}, \psi \in J_{k_2,m_2}$, then $[\phi, \psi]_\nu \in J_{k_1+k_2+2\nu, m_1+m_2}$.
- $[\phi, \psi]_\nu \in J^{cusp}_{k_1+k_2+2\nu, m_1+m_2}$ for $\nu > 0$.

# Special values of $L$- series

$$\phi(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z},\\ 4nm>r^2}} c_\phi(n,r)q^n\zeta^r\in J^{cusp}_{*,*},\quad \psi(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z},\\ 4nm>r^2}} c_\psi(n,r)q^n\zeta^r\in J^{cusp}_{k_1,m_1}.$$

- **Choie and Kohnen (1997):**

$$\langle\phi,[E_{k_2,m_2},\psi]_\nu\rangle = C\sum_{\substack{n,r\in\mathbb{Z}\\ 4m_1n>r^2}}\frac{(4m_1n-r^2)^\nu c_\phi(n,r)\overline{c_\psi(n,r)}}{(4(m_1+m_2)-r^2)^{k-3/2}}.$$

- **Sakata (1998):**

$$\langle\phi,\psi P^{(n,r)}_{k,m}]_\nu\rangle = C\sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0\\ 4(m_1+m_2)(n+n_1)-(r+r_1)^2>0}}\frac{(4m_2n_1-r_1^2)^{\nu-l}\overline{c_\phi(n_1,r_1)}c_\psi(n+n_1,r+r_1)}{(4(n+n_1)(m_1+m_2)-(r+r_1)^2)^{k_1+k_2+2\nu-\frac{3}{2}}}$$

- **–, Sahu(2016):** $\langle\phi,[P_{k,m},\psi]_\nu\rangle=$

$$C\sum_{l=0}^{\nu}A_l(k_1,m_1,k_2,m_2;\nu)\sum_{\substack{n_1,r_1\in\mathbb{Z}\\ 4m_2n_1-r_1^2>0\\ 4(m_1+m_2)(n+n_1)-(r+r_1)^2>0}}\frac{(4m_2n_1-r_1^2)^{\nu-l}\overline{c_\phi(n_1,r_1)}c_\psi(n+n_1,r+r_1)}{(4(n+n_1)(m_1+m_2)-(r+r_1)^2)^{k_1+k_2+2\nu-\frac{3}{2}}}$$

# Result

**Theorem (–, Pandey)**

Let $k_1, k_2, m_1$ and $m_2$ be positive integers. Let $\phi \in J_{k_1,m_1}$ and $h \in J_{k_1+k_2+2\nu,m_1+m_2}$ be non-constant Jacobi forms. Then each solution $\psi$ of the following differential equation

$$\sum_{r=0}^{\nu}(-1)^r \binom{k_1+\nu-\frac{3}{2}}{\nu-r}\binom{k_2+\nu-\frac{3}{2}}{r} m_1^{\nu-r} m_2^r L_{m_1}^r(\phi) L_{m_2}^{\nu-r}(\psi) = h$$

satisfies the transformation properties of a Jacobi form of weight $k_2$ and index $m_2$.

# Result

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- $\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z},\\ 4nm>r^2}} c_\phi(n,r) q^n \zeta^r.$
- **Generalized Poincaré series associated to $\phi$ :**

$$P_{k,m}(\phi) := \sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J} \phi|_{k,m}\gamma.$$

- $\mathbb{P}_{k,m}(1) = E_{k,m},\ \ \mathbb{P}_{k,m}(e^{2\pi i(n\tau+rz)}) = P_{k,m}^{(n,r)}.$

# Jacobi Poincaré series and Rankin-Cohen brackets

## Theorem (–,Pandey)

Let $k_1, k_2(\geq 11)$, $m_1$, $m_2$ and $\nu$ be positive integers. Let $N, R \in \mathbb{Z}$ be such that $4Nm_2 > R^2$. Consider the function $\psi(\tau, z)$ defined by

$$\begin{aligned}\psi(\tau, z) &= q^N \zeta^R \sum_{r=0}^{\nu} (-1)^r \binom{k_1 + \nu - \frac{3}{2}}{\nu - r} \binom{k_2 + \nu - \frac{3}{2}}{r} m_1^{\nu - r} m_2^r \\ &\times (4Nm_2 - R^2)^{\nu - r} L_{m_1}^r(\phi),\end{aligned}$$

where $\phi \in J_{k_1, m_1}$ (with $k_2 \geq k_1 + 10$ when $\phi$ is not a cusp form). Then we have

$$[\phi, P_{k_2, m_2}^{(N,R)}]_\nu = \mathbb{P}_{k_1 + k_2 + 2\nu, m_1 + m_2}(\psi).$$

## Theorem (–, Pandey)

Let $\phi$ be a Jacobi form of weight $k_1$, index $m_1$, and $\psi$ be a formal $(q, \zeta)$-series such that $\mathbb{P}_{k_2, m_2}(\psi)$ is well defined. Assume that $k_2 \geq k_1 + 2$ when $\phi$ is not a cusp form. Then

$$[\phi, \mathbb{P}_{k_2, m_2}(\psi)]_\nu = \mathbb{P}_{k_1 + k_2 + 2\nu, m_1 + m_2}([\phi, \psi]_\nu).$$

# Results

**Lemma**

Let $\phi$ be a complex-valued holomorphic function defined on $\mathbb{H} \times \mathbb{C}$. Then for a non-negative integer $\nu$ and $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have

$$L_m^{\nu}(\phi)|_{k+2\nu,m}M = \sum_{l=0}^{\nu} \binom{\nu}{l} \left(\frac{2mc}{\pi i}\right)^{\nu-l} \frac{(k+\nu-\frac{3}{2})!}{(k+l-\frac{3}{2})!} \frac{L_m^{l}(\phi|_{k,m}M)}{(c\tau+d)^{\nu-l}}.$$

**Lemma**

Let $\phi$ be a Jacobi form of weight $k$ and index $m$. Then for a non-negative integer $\nu$ and $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have

$$L_m^{\nu}(\phi|_{k,m}M) = \sum_{l=0}^{\nu} (-1)^{\nu-l} \binom{\nu}{l} \left(\frac{2mc}{\pi i}\right)^{\nu-l} \frac{(k+\nu-\frac{3}{2})!}{(k+l-\frac{3}{2})!} \frac{L_m^{l}(\phi)|_{k+2l,m}M}{(c\tau+d)^{\nu-l}}.$$

# Thank you!