Double Eisenstein Series and Modular Forms of Level 4

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Multiple Zeta Values

Definition

For integers $k_1, \ldots, k_{r-1} \geq 1$ and $k_r \geq 2$, we define the multiple zeta value by

$$\zeta(k_1, \ldots, k_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

Especially, $\zeta(k_1, k_2)$ is called the double zeta value.

Gangle–Kaneko–Zagier considered the $\mathbb{Q}$-vector space generated by double zeta values:

$$\mathcal{DZ}_k := \langle \zeta(k-r, r) \mid 2 \leq r \leq k-1 \rangle_{\mathbb{Q}}.$$

Furthermore, they revealed the relationship between double zeta values and modular forms by introducing the double Eisenstein series.

Double Eisenstein Series

Definition [Gangl–Kaneko–Zagier]

For integers $r \geq 1$ and $s \geq 2$, we define the double Eisenstein series by

$$G_{r,s}(\tau) := \sum_{0 \prec m_1\tau+n_1 \prec m_2\tau+n_2} \frac{1}{(m_1\tau+n_1)^r(m_2\tau+n_2)^s}$$

on the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. Here,

$$0 \prec m\tau + n \overset{\text{def}}{\iff} \begin{cases} m = 0 \ \text{ and } \ n > 0 \\ \text{or} \\ m > 0 \end{cases},$$

$$m_1\tau + n_1 \prec m_2\tau + n_2 \overset{\text{def}}{\iff} 0 \prec (m_2 - m_1)\tau + (n_2 - n_1).$$

- $G_{r,s}(\tau + 1) = G_{r,s}(\tau) \rightsquigarrow$ Fourier expansion $\rightsquigarrow G_{r,s}(\tau) = \zeta(r,s) + O(e^{2\pi i\tau})$.

Previous Work

Let

$$\mathcal{DE}_k := \langle G_{k-r,r} \mid 2 \le r \le k-1 \rangle_{\mathbb{Q}}$$
$$\mathcal{DZ}_k := \langle \zeta(k-r,r) \mid 2 \le r \le k-1 \rangle_{\mathbb{Q}}$$

Theorem [Gangl–Kaneko–Zagier, Kaneko]

For any integer $k \ge 3$, we have

$$\mathcal{DE}_k \supset \mathbb{Q}G_k \oplus \mathbb{Q}(2\pi i)^2 G'_{k-2} \oplus (2\pi i)^k S_k^{\mathbb{Q}}(SL_2(\mathbb{Z})) \tag{1}$$
$$\dim_{\mathbb{Q}} \mathcal{DE}_k = [(k+1)/2] \tag{2}$$
$$\dim_{\mathbb{Q}} \mathcal{DZ}_k \le [(k+1)/2] - 1 - \dim_{\mathbb{Q}} S_k^{\mathbb{Q}}(SL_2(\mathbb{Z})) \tag{3}$$

- The statements (1) and (2) lead to the inequality (3) by $G_{r,s}(\tau) = \zeta(r,s) + O(e^{2\pi i \tau})$.
- We conjecture that the inequality (3) holds as an equality. [Zagier]
  $\rightsquigarrow$We can expect that the $\mathbb{Q}$-linear equations among double zeta values all come from modular forms.

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Multiple $\widetilde{T}$-values

Definition [Kaneko–Tsumura]

For integers $k_1, \ldots, k_r \geq 1$, we define multiple $\widetilde{T}$-value by

$$\widetilde{T}(k_1, \ldots, k_r) := \sum_{0<n_1<\cdots<n_r} \frac{\chi_4(n_1)\chi_4(n_2-n_1)\cdots\chi_4(n_r-n_{r-1})}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}},$$

where χ_4 is the non-trivial Dirichlet character modulo 4.

- Multiple $\widetilde{T}$-values are objects that can be referred to as multiple zeta values of level 4.

We consider the $\mathbb{Q}$-vector space generated by double $\widetilde{T}$-values:

$$\mathcal{D}\widetilde{\mathcal{T}}_k := \langle \widetilde{T}(k-r, r) \mid 1 \leq r \leq k-1 \rangle_{\mathbb{Q}}.$$

Furthermore, we observe the relationship between double $\widetilde{T}$-values and modular forms on $\Gamma_0(4)$ by using the double $\widetilde{H}$-series.

Double $\widetilde{H}$-series

Definition [Tasaka]

For integers $r, s \geq 1$, we define $\widetilde{H}_{r,s}$ by

$$\widetilde{H}_{r,s}(\tau) := \sum_{0 \prec 4m_1\tau+n_1 \prec 4m_2\tau+n_2} \frac{\chi_4(n_1)\chi_4(n_2-n_1)}{(4m_1\tau+n_1)^r(4m_2\tau+n_2)^s}.$$

- Double $\widetilde{H}$-series are objects that can be referred to as double Eisenstein series of level 4.
- The constant term of Fourier series of $\widetilde{H}_{r,s}(\tau)$ is $\widetilde{T}(r, s)$.

Main results

$$\mathcal{D}\widetilde{\mathcal{H}}_k := \langle \widetilde{H}_{k-r,r} \mid 1 \le r \le k-1 \rangle_{\mathbb{Q}},$$
$$\mathcal{D}\widetilde{\mathcal{T}}_k := \langle \widetilde{T}(k-r,r) \mid 1 \le r \le k-1 \rangle_{\mathbb{Q}},$$
$$\mathcal{S}\mathcal{D}\widetilde{\mathcal{H}}_k := \mathcal{D}\widetilde{\mathcal{H}}_k \cap (2\pi i)^k S_k^{\mathbb{Q}}(\Gamma_0(4)).$$

Theorem [K., conjectured by Tasaka] (arXiv:2403.06917)

For any integer $k \ge 2$, we have

$$\dim_{\mathbb{Q}} \mathcal{D}\widetilde{\mathcal{H}}_k = k-1,$$
$$\dim_{\mathbb{Q}} \mathcal{S}\mathcal{D}\widetilde{\mathcal{H}}_k = \begin{cases} \left[\frac{k-2}{4}\right] & (k : \text{even}) \\ 0 & (k : \text{odd}) \end{cases},$$
$$\dim_{\mathbb{Q}} \mathcal{D}\widetilde{\mathcal{T}}_k \le k-1-\dim_{\mathbb{Q}} \mathcal{S}\mathcal{D}\widetilde{\mathcal{H}}_k. \tag{4}$$

- We conjecture that the inequality (4) holds as an equality. [Kaneko–Tsumura]
  $\rightsquigarrow$We can expect that the $\mathbb{Q}$-linear equations among double $\widetilde{T}$-values all come from modular forms.

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Outline of the proof

In the following, we assume $k \geq 6$ is even.

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1. $\widetilde{H}_{k-r}\widetilde{H}_r \in \mathcal{D}\widetilde{\mathcal{H}}_k \cap (2\pi i)^k M_k^{\mathbb{Q}}(\Gamma_0(4))$ for odd integer $3 \leq r \leq k-3$. Here,

$$\widetilde{H}_r(\tau) := \sum_{0 \prec 4m\tau+n} \frac{\chi_4(n)}{(4m\tau+n)^r}.$$

Proof) Calculations using the Fourier series of $\widetilde{H}_r$.

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2. $B_{M_k} := \{\widetilde{H}_{k-r}\widetilde{H}_r \mid 3 \leq r \leq k-3 : \text{odd}\}$ is independent. $(\#B_{M_k} = [(k-2)/4])$

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2. $B_{M_k} := \{\widetilde{H}_{k-r}\widetilde{H}_r \mid 3 \leq r \leq k-3 : \text{odd}\}$ is independent. $(\#B_{M_k} = [(k-2)/4])$

$$\rightsquigarrow \text{We can obtain a basis for } S\mathcal{D}\widetilde{\mathcal{H}}_k \text{ from } B_{M_k}.$$

Is $B_{M_k} = \{\widetilde{H}_{k-r}\widetilde{H}_r \mid 3 \le r \le k-3 : \text{odd}\}$ independent?

We transform the claim to be proven. (cf. [Kohnen–Zagier, Antoniadis, Fukuhara–Yang])

Lemma

If $f \in S_k(\Gamma_0(4))$ is a normalized Hecke eigenform, then

$$\langle \widetilde{H}_{k-r}\widetilde{H}_r, f\rangle_{\Gamma_0(4)} = \rho_{k,r} L(f, k-1) L(f_{\chi_4}, r) \quad (3 \le r \le k-3 : \text{odd}).$$

Let

$$B_k := \{f \in S_k(\Gamma_0(4)) \mid f \text{ is a newform of level 1 or 2 or 4}\},$$

and we label the functions of B_k by $f_1, \ldots, f_d$, where $d = \#B_k = [(k-2)/4]$ $(= \#B_{M_k})$.

Lemma

$$B_{M_k} \text{ is independent} \Leftrightarrow \det\big(\langle \widetilde{H}_{k-(2j-1)}\widetilde{H}_{2j-1}, f_i\rangle_{\Gamma_0(4)}\big)_{1\le i,j\le d} \ne 0$$

$$\Leftrightarrow \det\big(L(f_{i,\chi_4}, 2j+1)\big)_{1\le i,j\le d} \ne 0.$$

$\det\bigl(L(f_{i,\chi_4}, 2j+1)\bigr)_{1\le i,j\le d} \neq 0$?

For $f(\tau) \in S_k(\Gamma_0(4))$, we consider n-period of f

$$r_n(f) := \int_0^{i\infty} f(\tau)\tau^n d\tau = \frac{n!}{(-2\pi i)^{n+1}} L(f, n+1).$$

Since the map $S_k(\Gamma_0(4)) \ni f \mapsto r_n(f_{\chi_4}) \in \mathbb{C}$ is linear, there exists a unique cusp form $R_{k,n} \in S_k(\Gamma_0(4))$ such that

$$r_n(f_{\chi_4}) = \langle f, R_{k,n}\rangle_{\Gamma_0(4)}.$$

Lemma

Let $A_k := \bigl(\langle R_{k,2i-1}, R_{k,2j}\rangle_{\Gamma_0(4)}\bigr)_{1\le i,j\le d} = \bigl(r_{2j}(R_{k,2i-1,\chi_4})\bigr)_{1\le i,j\le d}$, then

$$\det A_k \neq 0 \;\Rightarrow\; \det\bigl(\langle f_i, R_{2j}\rangle_{\Gamma_0(4)}\bigr)_{1\le i,j\le d} \neq 0 \;\Leftrightarrow\; \det\bigl(L(f_{i,\chi_4}, 2j+1)\bigr)_{1\le i,j\le d} \neq 0.$$

$$\det A_k \left(= \left(\langle R_{k,2i-1}, R_{k,2j} \rangle_{\Gamma_0(4)} \right)_{1 \le i,j \le d} = \left(r_{2j}(R_{k,2i-1,\chi_4}) \right)_{1 \le i,j \le d} \right) \neq 0 \ ?$$

We can specifically identify $R_{k,n}$.

Lemma

For any integer n with $0 < n < k-2$,

$$R_{k,n}(\tau) = \alpha_{k,n} \sum_{\gamma \in \Gamma_0(4)} \frac{1}{\tau^{k-n-1}} \bigg|_k \left(\begin{pmatrix} 1 & 1/4 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1/4 \\ 0 & 1 \end{pmatrix} \right) \gamma.$$

Therefore, we can specifically identify $r_{2j}(R_{k,2i-1,\chi_4})$, and so we can obtain a concrete representation of A_k.

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Therefore, we can specifically identify $r_{2j}(R_{k,2i-1,\chi_4})$, and so we can obtain a concrete representation of A_k.

Proposition

$\det A_k \neq 0$.

This completes the proof.

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Future Work

- Similar results have been obtained for level 2 by Kaneko–Tasaka.

Question

To what extent do the relations among multiple (not necessary double) zeta values of level N come from modular forms ?

Thank you for your attention!

Main results

- $\widetilde{H}_k(\tau) := \sum_{0 \prec 4m\tau+n} \frac{\chi_4(n)}{(4m\tau+n)^k} \in M_k(\Gamma_0(4), \chi_4)$ for an odd $k \geq 3$.
- $\widetilde{G}_k(\tau) := \sum_{0 \prec 4m\tau+n} \frac{\chi_0(n)}{(4m\tau+n)^k} \in M_k(\Gamma_0(4))$ for an even $k \geq 4$, where χ_0 is the trivial Dirichlet character of modulo 4.

Theorem [K.]

$$\widetilde{H}_{k_1}(\tau)\widetilde{H}_{k_2}(\tau) = \sum_{p=1}^{k_1+k_2-1} \left(\binom{p-1}{k_1-1} + \binom{p-1}{k_2-1} \right) \widetilde{H}_{k_1+k_2-p,p}(\tau)$$

$$\widetilde{G}_k(\tau) = \frac{1}{k-1} \left(\sum_{p=1}^{k-1} 2^{k-2-p} \widetilde{H}_{p,k-p}(\tau) + \frac{1}{2} \widetilde{H}_{k-1,1}(\tau) \right)$$

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- $\widetilde{H}_k(\tau) := \sum_{0 \prec 4m\tau+n} \frac{\chi_4(n)}{(4m\tau+n)^k} \in M_k(\Gamma_0(4), \chi_4)$ for an odd $k \geq 3$.
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Theorem [K.]

The set $\{\widetilde{G}_k, \widetilde{H}_r\widetilde{H}_{k-r} \mid 3 \leq r \leq k-3 : \text{odd}\}$ is a basis of $M\mathcal{D}\widetilde{\mathcal{H}}_k (:= \mathcal{D}\widetilde{\mathcal{H}}_k \cap (2\pi i)^k M_k^{\mathbb{Q}}(\Gamma_0(4)))$.

Main results

Theorem [K.]

For any even integers $k \geq 6$ and $1 \leq j \leq [(k-2)/4]$, let

$$a^j_{k,p} := \frac{B_k}{k}\left(\binom{k-p-1}{r-1} + \binom{k-p-1}{k-r-1}\right) - \frac{(1+\delta_{1,p})}{2^{p+1}}\frac{1}{2^k-1}\binom{k-2}{r-1}E_{r-1}E_{k-r-1} \quad (r = 2j+1).$$

Then we have

$$\sum_{p=1}^{k-1} a^j_{k,p}\widetilde{T}(p, k-p) = 0.$$

Moreover, when k is fixed, $[(k-2)/4]$ equations are independent.

Is $B_{S_k} := \{\widetilde{H}_{k-r}\widetilde{H}_r \mid 3 \le r \le k-3 : \text{odd}\}$ independent?

We transform the claim to be proven. (c.f. [Kohnen-Zagier, Antoniadis, Fukuhara-Yang])

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If $f \in S_k(\Gamma_0(4))$ is a normalized Hecke eigenform, then

$$\langle \widetilde{H}_{k-r}\widetilde{H}_r, f\rangle_{\Gamma_0(4)} = \rho_k(r)L(f, k-1)L(f_{\chi_4}, r) \quad (3 \le r \le k-3 : \text{odd}).$$

For $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(4))$,

$$L(f, r) = \sum_{n=1}^{\infty} \frac{a_n}{n^r}, \quad L(f_{\chi_4}, r) = \sum_{n=1}^{\infty} \frac{a_n \chi_4(n)}{n^r}.$$

Here, the values of L-functions at small integers are defined by using analytic continuation.