Dimension Formulas for Siegel Modular Forms of Level 4

Ralf Schmidt, UNT

joint work with Manami Roy (Lafayette College)
and Shaoyun Yi (Xiamen University)

May 22, 2024

Dimension Formulas for Classical Modular Forms

Since $\bigoplus_{k=0}^{\infty} M_k(\mathrm{SL}(2,\mathbb{Z})) = \mathbb{C}[E_4, E_6]$, we have

$$\sum_{k=0}^{\infty} \dim M_k(\mathrm{SL}(2,\mathbb{Z}))t^k = \frac{1}{(1-t^4)(1-t^6)}.$$

Multiplication by Δ gives $M_k(\mathrm{SL}(2,\mathbb{Z})) \cong S_{k+12}(\mathrm{SL}(2,\mathbb{Z}))$, and hence

$$\sum_{k=0}^{\infty} \dim S_k(\mathrm{SL}(2,\mathbb{Z}))t^k = \frac{t^{12}}{(1-t^4)(1-t^6)}.$$

Other examples:

$$\sum_{k=0}^{\infty} \dim S_k(\Gamma_0(2))t^k = \frac{t^8}{(1-t^2)(1-t^4)},$$

$$\sum_{k=0}^{\infty} \dim S_k(\Gamma_0(4))t^k = \frac{t^6}{(1-t^2)^2}.$$

Siegel Modular Forms (degree 2)

- Siegel upper half space:

$$\mathbb{H}_2 = \{Z \in M(2 \times 2, \mathbb{C}) \mid {}^tZ = Z,\ \mathrm{im}(Z) > 0\}.$$

- $f : \mathbb{H}_2 \to \mathbb{C}$ holomorphic with

$$f\left((AZ+B)(CZ+D)^{-1}\right) = \det(CZ+D)^k f(Z) \quad \text{for } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma.$$

There are many choices for Γ, for example:

- The full modular group $\mathrm{Sp}(4, \mathbb{Z})$
- The principal congruence subgroup of level N:

$$\Gamma(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} 1+N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & 1+N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & 1+N\mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & 1+N\mathbb{Z} \end{bmatrix}$$

More Γ's

- The Siegel congruence subgroup of level N:

$$\Gamma_0(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

- The Klingen congruence group of level N:

$$\Gamma_0'(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

- The Borel congruence subgroup of level N:

$$B(N) = \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

- The paramodular group of level N:

$$K(N) = \mathrm{Sp}(4,\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

(Looks nicer after switching first row and column.)

Classical Results for Siegel Modular Forms

Igusa (1964):

$$\sum_{k=0}^{\infty} \dim M_k(\mathrm{Sp}(4,\mathbb{Z}))t^k = \frac{1}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

$$\sum_{k=0}^{\infty} \dim S_k(\mathrm{Sp}(4,\mathbb{Z}))t^k = \frac{t^{10}+t^{12}-t^{22}+t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

Igusa can be used for $\dim M_k(\Gamma)$ whenever (a conjugate of) Γ lies between $\Gamma(2)$ and $\mathrm{Sp}(4,\mathbb{Z})$, for example

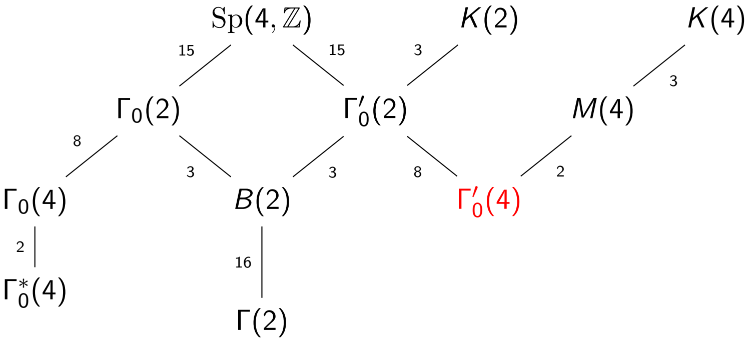
$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0(2))t^k = \frac{1+t^{19}}{(1-t^2)(1-t^4)^2(1-t^6)},$$

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0(4))t^k = \frac{1+t^4+t^{11}+t^{15}}{(1-t^2)^3(1-t^6)}.$$

Known Cases for dim $S_k(\Gamma)$

$\Gamma(N)$	all N	Morita (1974), Yamazaki (1976)
	all N, vector valued case	Tsushima (1983), Wakatsuki (2012)
$\Gamma_0(N)$	$N = p$	Hashimoto (1983)
	$N = p,$ vector valued case	Tsushima (1997)
	$N = 4$	Tsushima (2003)
$\Gamma_0'(N)$	$N = p$	Hashimoto, Ibukiyama (1985)
	$N = p,$ vector valued case	Wakatsuki (2013)
	$N = 4$	Roy, S, Yi (2023)
$K(N)$	$N = 2$	Ibukiyama, Onodera (1997)
	square-free N, vv case	Ibukiyama, Kitayama (2017)
	$N = 4$	Poor, Yuen (2012)
	$N = 8, 16, \ k \leq 14$	Poor, S, Yuen (2018)
$B(N)$	$N = p$	Hashimoto, Ibukiyama (1985)
	$N = p,$ vector valued case	Wakatsuki (2013)

Some Congruence Subgroups of Level 1, 2 or 4



$$\Gamma_0^*(4) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_0(4) : D \bmod 2 \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\} \right\}$$

Automorphic Representations

Basic Principle: Every cusp form arises as a special vector inside an automorphic representation.

- $\pi \cong \bigotimes_{p \leq \infty} \pi_p$ cuspidal, automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$.
- (π_p, V_p) irreducible, admissible representation of $\mathrm{GSp}(4, \mathbb{Q}_p)$.
- Choose $v_p \in V_p$ for all p (spherical vector almost everywhere).
- Then $v = \otimes v_p$ corresponds to an automorphic form $\Phi : \mathrm{GSp}(4, \mathbb{A}) \to \mathbb{C}$.
- Φ "descends" to a function $F : \mathbb{H}_2 \to \mathbb{C}$, which transforms like a Siegel modular form.
- If π_∞ and v_∞ are chosen correctly, then $F \in S_k(\Gamma)$ for some Γ.
- Γ depends on the choice of the v_p.

The same π can give rise to many F's.

Borel-induced Representations of $\mathrm{GSp}(4, \mathbb{Q}_p)$

	constituents of		representation	tempered	L^2	g
I	$\chi_1 \times \chi_2 \rtimes \sigma$		(irreducible)	χ_i, σ unit.		•
II	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	χ, σ unit.		•
	$(\chi^2 \neq \nu^{\pm 1}, \chi \neq \nu^{\pm 3/2})$	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$			
III	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	π, σ unit.		•
	$(\chi \notin \{1, \nu^{\pm 2}\})$	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$			
IV	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	σ unit.	•	•
		b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$			
		c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$			
		d	$\sigma 1_{\mathrm{GSp}(4)}$			
V	$\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	σ unit.	•	•
	$(\xi^2 = 1,\ \xi \neq 1)$	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$			
		c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$			
		d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$			
VI	$\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$	a	$\tau(S, \nu^{-1/2}\sigma)$	σ unit.		•
		b	$\tau(T, \nu^{-1/2}\sigma)$	σ unit.		
		c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$			
		d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$			

The Local Dimensions $d_{\Gamma,\Omega}$ for $\mathrm{GSp}(4,\mathbb{Q}_2)$

Ω		$\Gamma(\mathfrak{p})$	K	$K(\mathfrak{p})$	$K(\mathfrak{p}^2)$	$\Gamma_0(\mathfrak{p})$	$\Gamma_0(\mathfrak{p}^2)$	$\Gamma_0^*(\mathfrak{p}^2)$	$\Gamma_0'(\mathfrak{p})$	$\Gamma_0'(\mathfrak{p}^2)$	$M(\mathfrak{p}^2)$	$B(\mathfrak{p})$
I		45	1	2	4	4	12	15	4	11	8	8
II	a	30	0	1	2	1	5	8	2	7	5	4
	b	15	1	1	2	3	7	7	2	4	3	4
III	a	30	0	0	1	2	8	10	1	5	3	4
	b	15	1	2	3	2	4	5	3	6	5	4
IV	a	16	0	0	0	0	2	4	0	2	1	1
	b	14	0	0	1	2	6	6	1	3	2	3
	c	14	0	1	2	1	3	4	2	5	4	3
	d	1	1	1	1	1	1	1	1	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
XI	a	10	0	0	0	0	1	4	0	2	1	0
	b	5	0	0	1	0	0	3	0	1	1	0
Va*		1	0	0	0	0	0	1	0	0	0	0
sc(16)		9	0	0	0	0	0	3	0	1	0	0

Dimensions and Automorphic Representations Count

Definition

Let k be a positive integer. Let $A_k(\Omega)$ be the set of cuspidal automorphic representations $\pi \cong \otimes_p \pi_p$ of $\mathrm{GSp}(4, \mathbb{A})$ with the following properties:

1. π has trivial central character.
2. π_∞ is the lowest weight module with minimal K-type (k, k).
3. π_p is unramified for each finite $p \neq 2$.
4. π_2 is an irreducible, admissible representation of $\mathrm{GSp}(4, \mathbb{Q}_2)$ of type Ω with non-trivial $\Gamma(\mathfrak{p})$-invariant vectors.

Then

$$\dim S_k(\Gamma) = \sum_{\Omega} |A_k(\Omega)| \cdot d_{\Gamma,\Omega} \qquad (\star)$$

The Idea

$$\dim S_k(\Gamma) = \sum_{\Omega} |A_k(\Omega)| \cdot d_{\Gamma,\Omega} \qquad (\star)$$

(1) Use $(\star)$ for Γ's for which LHS is known.

(2) Hope you have enough Γ's to be able to determine the $|A_k(\Omega)|$.

(3) Then use $(\star)$ again for Γ's for which LHS was previously unknown.

This works for:

- Known cases: $\mathrm{Sp}(4,\mathbb{Z})$, $\Gamma(2)$, $\Gamma_0(2)$, $\Gamma_0(4)$, $\Gamma_0'(2)$, $B(2)$, $K(2)$, $K(4)$
- Known after some work: $\Gamma_0^*(4)$, $M(4)$
- Previously unknown case: $\Gamma_0'(4)$

It's a bit more complicated

The cuspidal, automorphic representations of $\mathrm{GSp}(4, \mathbb{A})$ come in six **Arthur types**:

		contributes to $A_k(\Gamma)$?
(G)	general type	yes
(Y)	Yoshida type	no
(P)	Saito - Kurokawa type	yes
(Q)	Soudry type	no
(B)	Howe - Piatetski-Shapiro type	no
(F)	one-dimensional representations	no

It all comes down to this 10×10 matrix being invertible:

$$\begin{bmatrix} 45 & 1 & 2 & 4 & 4 & 12 & 15 & 4 & 8 & 8 \\ 30 & 0 & 1 & 2 & 1 & 5 & 8 & 2 & 5 & 4 \\ 30 & 0 & 0 & 1 & 2 & 8 & 10 & 1 & 3 & 4 \\ 16 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 1 & 1 \\ 22 & 0 & 0 & 1 & 0 & 2 & 6 & 1 & 3 & 2 \\ 15 & 0 & 0 & 0 & 0 & 4 & 5 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 3 & 4 & 0 & 0 & 0 \\ 15 & 0 & 0 & 1 & 0 & 1 & 7 & 0 & 2 & 0 \\ 10 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 1 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}$$

Final Result

$$\sum_{k=0}^{\infty} \dim S_k(\Gamma_0'(4))t^k$$
$$= \frac{t^7(1+3t+2t^2+9t^3+5t^4+13t^5+4t^6+6t^7+5t^8+4t^{10}-3t^{11}+2t^{12}-2t^{13}-2t^{15}+t^{16})}{(1-t^4)^2(1-t^6)^2}$$

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0'(4))t^k$$
$$= \frac{1+2t^4+4t^6+t^7+5t^8+2t^9+4t^{10}+5t^{11}+5t^{12}+4t^{13}+2t^{14}+5t^{15}+t^{16}+4t^{17}+2t^{19}+t^{23}}{(1-t^4)^2(1-t^6)^2}$$

Observation: The numerator polynomial for $M_k(\Gamma_0'(4))$ is palindromic.

(Some low weight cases were calculated by Cris Poor and David Yuen.)

Thank You!