Congruences modulo prime powers for a class of partition functions

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1. Motivation
2. Background
3. Main Results
   - Example 1
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4. Sketch of Proofs

Partitions

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Euler Generating Function

$$\sum_{n=0}^{\infty} p(n) q^{n-\frac{1}{24}} = q^{\frac{-1}{24}} \prod_{m=1}^{\infty} \frac{1}{1-q^m} = \frac{1}{\eta(z)}.$$

Study of congruences for $p(n)$

Theorem (Ramanujan, 1919)

For $n \in \mathbb{N}_0$, we have:

$$p(an+b) \equiv 0 \pmod{m}$$

where $(a, b, m) = \{(5,4,5), (7,5,7), (11,6,11)\}$.

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Theorem (Ono, 2000)

Let $m \geqslant 5$, prime and let $k \geqslant 1$, integer. Then, a positive proportion of primes ℓ satisfy

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Theorem (Ahlgren, 2000)

Extension of (2) *to composite* $m, (m, 6) = 1$.

Results of Ahlgren - Boylan and Yang on $p(n)$

Theorem (Ahlgren-Boylan, 2003)

Let $\ell \geqslant 5$ be a prime and $j \geqslant 1$ be an integer and consider $1 \leqslant \beta_{\ell,j} \leqslant \ell^j - 1$ satisfying $24\beta_{\ell,j} \equiv 1 \pmod{\ell^j}$. Then there exists a modular form $F_{\ell,j}(z) \in M_{k_{\ell,j}}(\Gamma_0(1)) \cap \mathbb{Z}[[q]]$ such that

$$\sum_{n=0}^{\infty} p(\ell^j n + \beta_{\ell,j}) q^{n+\frac{r_{\ell,j}}{24}} \equiv \eta(z)^{r_{\ell,j}} F_{\ell,j}(z) \pmod{\ell^j}.$$

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Theorem (Yang, 2010)

Let $\ell \neq m$ be primes with $\ell \geqslant 13$ and $m \geqslant 5$, and let $j \geqslant 1$ be an integer. Then there exists $K \geqslant 2$ such that for all $u, n \geqslant 1$ with $m \nmid n$,

$$p\left(\frac{\ell^j m^{2uK-1} n + 1}{24}\right) \equiv 0 \pmod{\ell}.$$

Framework

We consider the partition function $p_{[1,p]}(n)$ for any prime p.

$$\sum_{n=0}^{\infty} p_{[1,p]}(n)q^n = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)(1-q^{pj})}$$

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Goal:

1. Prove analogues of Ahlgren - Boylan and Yang's result for the partition function $p_{[1,p]}(n)$ where p is prime.
2. Provide explicit numerical examples of our theorems.

Modular Forms

Definition (Modular Form)

A function $f : \mathcal{H} \to \mathbb{C}$ holomorphic is modular of weight $k \in \mathbb{Z}$ if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

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Growth at Cusps:

- f has linear exponential growth at cusps, $f \in \mathrm{M}_k^!(\Gamma_0(N)), \chi)$ (Weakly holomorphic)
- f is bounded at cusps, $f \in \mathrm{M}_k(\Gamma_0(N)), \chi)$ (Holomorphic)
- f vanishes at cusps, $f \in \mathrm{S}_k(\Gamma_0(N)), \chi)$ (Cusp Form)

Eta-Quotients

Definition

An *eta-quotient* of level $N \geqslant 1$ is a function of the form,

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta}, \text{ where } \delta, r_\delta \in \mathbb{Z} \text{ with } \delta \geqslant 1.$$

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Orders of vanishing of Eta-quotients

Let $c, d, N \in \mathbb{N}$ with $(c, d) = 1$, and let $h = \frac{N}{\gcd(d^2, N)}$. If $f(z)$ is an eta-quotient,

$$ord_{\frac{c}{d}}(f(z)) = \frac{N}{24\gcd(d^2, N)} \sum_{\delta | N} \frac{\gcd(d, \delta)^2}{\delta} r_\delta.$$

Operators on Modular Forms

Definition (Hecke Operator)

For all integers $n \geqslant 1$, we define the Hecke operator T_n on $M_k(\Gamma_0(N), \psi)$ by

$$T_n = \sum_{d|n} \psi(d) d^{k-1} V_d \circ U_{n/d}.$$

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Definition (Trace Operator)

Let ℓ be a prime and $N \geqslant 1$ such that $\ell \parallel N$. We define the trace operator as follows:

$$Tr^N_{N/\ell} : \mathrm{M}_k(\Gamma_0(N), \psi) \to \mathrm{M}_k(\Gamma_0(N/\ell), \psi),$$
$$Tr^N_{N/\ell}(f) = f + \psi(\ell)\ell^{1-k/2} f \mid_k W^N_\ell \mid U_\ell.$$

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Proposition (U - operator)

Let ℓ be prime with $\ell \mid N$. Further, if $\ell^2 \mid N$, then $U_\ell : M_k(\Gamma_0(N), \psi) \to M_k(\Gamma_0(N/\ell), \psi)$.

Theorem (Boylan-S.)

Let $\ell \neq p$ be primes with $\ell \geqslant 5$ and let $j \in \mathbb{Z}_{\geqslant 1}$. Then, there exists a modular form $H(z) \in M_k(\Gamma_0(p), \chi) \cap \mathbb{Z}[[q]]$ such that

$$\sum_{n=0}^{\infty} p_{[1,p]}(\ell^j n + \beta) q^{n+x} \equiv (\eta(z)\eta(pz))^y H(z) \pmod{\ell^j}.$$

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Remark

When $p + 1 \mid 24$,

$$\sum_{n=0}^{\infty} p_{[1,p]}\left(\frac{\ell^j n + 1}{D}\right) q^n \in \mathcal{A}_{p,y,k,\chi} \pmod{\ell^j},$$

$\mathrm{A}_{p,y,k,\chi} = \{(\eta(Dz)\eta(Dpz))^y \, H(Dz) : H(z) \in M_k(\Gamma_0(p), \chi)\}$.

Remark

1. $\mathcal{A}_{p,y,k,\chi} \cong M_k(\Gamma_0(p), \chi)$
2. For $y > 0, \mathcal{A}_{p,y,k,\chi} \subseteq M_{k+y}\left(\Gamma_0(pD^2), \left(\frac{-p}{\cdot}\right)\right)$

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Theorem (Boylan - Warnock)

Let $p \neq m$ primes with $p \in \{2, 3, 5\}$, $m \geqslant 5$. Then, we have

$$T_{m^2} : \mathcal{A}_{p,y,k,\chi} \to \mathcal{A}_{p,y,k,\chi}$$

Values of parameters for odd j

Conditions	$j \geqslant 1$ odd		
	x	y	k
$24 \mid p+1, D=1$ and $\frac{p+1}{\Delta} < \ell^j$	0	0	$\ell^{j-1}(\ell-1)-1$
$24 \mid p+1, D=1$ and $\frac{p+1}{\Delta} \geqslant \ell^j$	$-\left\lfloor \frac{p+1}{\Delta \ell^j} \right\rfloor$	-1	$\ell^{j-1}(\ell-1)$
$p+1 \mid 24, D=\frac{24}{p+1}$ or $p \in \{13,17,19\}$, $\ell \equiv D-1 \pmod{D}$ except for $(p,\ell,j)=(19,5,1)$	$\frac{p+1}{\Delta}\left(1+\left\lfloor \frac{\ell}{D} \right\rfloor - \frac{\ell}{D}\right)$	$D\left(1+\left\lfloor \frac{\ell}{D} \right\rfloor\right)-\ell$ $= D-r$ where $\ell = D\left\lfloor \frac{\ell}{D} \right\rfloor + r$, $0<r<D$	$(\ell-1)(\ell^{j-1}+1)$ $-D\left(1+\left\lfloor \frac{\ell}{D} \right\rfloor\right)$
$p+1 \nmid 24,\ 24 \nmid p+1$	$\frac{p+1}{\Delta}\left(\left\lfloor \frac{\ell}{D} \right\rfloor - \frac{\ell}{D}\right)+1$ $+\left(\left\lfloor \frac{p+1}{\Delta}\left(\frac{\ell}{D}-\left\lfloor \frac{\ell}{D} \right\rfloor - \frac{1}{D\ell^j}\right)-\frac{1}{\ell^j} \right\rfloor\right)$	$D\left\lfloor \frac{\ell}{D} \right\rfloor - \ell = -r$	$(\ell-1)(\ell^{j-1}+1)-D\left(\left\lfloor \frac{\ell}{D} \right\rfloor\right)$

Values of parameters for even j

Conditions	$j \geqslant 1$ even		
	x	y	k
$24 \mid p+1, D=1$ and $\frac{p+1}{\Delta} < \ell^j$	0	0	$\ell^{j-1}(\ell-1)-1$
$24 \mid p+1, D=1$ and $\frac{p+1}{\Delta} \geqslant \ell^j$	$-\left\lfloor \frac{p+1}{\Delta \ell^j} \right\rfloor$	-1	$\ell^{j-1}(\ell-1)$
$p+1 \mid 24,\ D=\frac{24}{p+1}$	$\frac{D-1}{D}$	$D-1$	$\ell^{j-1}(\ell-1)-D$
$p+1 \nmid 24,\ \ 24 \nmid p+1$	$1+\left\lfloor \frac{p+1}{24} - \left(\frac{\frac{p+1}{24}+1}{\ell^j} \right) \right\rfloor - \left(\frac{p+1}{24} \right)$	-1	$\ell^{j-1}(\ell-1)$

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- Basis for $M_4(\Gamma_0(5))$: $\left\{ f_0(z) = \frac{\eta(z)^{10}}{\eta(5z)^2},\ f_1(z) = \eta(z)^4\ \eta(5z)^4,\ f_2(z) = \frac{\eta(5z)^{10}}{\eta(z)^2} \right\}$.

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- For primes $\ell \in \{7, 11, 13, 17\}$, computations reveal that

$$\sum_{n=0}^{\infty} p_{[1,5]}\left(\frac{\ell n + 1}{4}\right) q^n \equiv (\eta(4z)\eta(20z))^y H(4z) \pmod{\ell},$$

with $H(z) \in M_k(\Gamma_0(5))$ and integers k and $y \geqslant 0$ as follows:

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with $H(z) \in M_k(\Gamma_0(5))$ and integers k and $y \geqslant 0$ as follows:

$H(z)$	ℓ	k	y
$2f_0 + f_1 + 5f_2$	7	4	1
$3f_0^2 + 7f_0f_1 + 3f_1^2 + 6f_1f_2 + 4f_2^2$	11	8	1
$12f_0^2 + 2f_0f_1 + 6f_1^2 + 3f_1f_2 + f_2^2$	13	8	3
$10f_0^3 + 16f_0^2f_1 + 7f_0^2f_2 + 13f_0f_1f_2 + 8f_0f_2^2 + 15f_1f_2^2 + f_2^3$	17	12	3

Analogue of Yang's result

- $\ell \neq m \neq p$ primes, $m \geqslant 5, p \in \{2,3,5\}$ and $j \geqslant 1$, integer

Theorem (Boylan - S.)

There exist explicitly computable integers $J > 2$ and $N \geqslant 1$ such that the following congruences hold.

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2. *For all non-negative integers w and n, we have*

$$p_{[1,p]}\left(\frac{\ell^j\ m^{w}\ n+1}{D}\right) \equiv p_{[1,p]}\left(\frac{\ell^j\ m^{2N+w}\ n+1}{D}\right) \pmod{\ell^j}.$$

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- B. - S. $\implies \exists\ H(z) \in \mathrm{M}_{36}(\Gamma_0(3))$,

$$\sum_{n=0}^{\infty} p_{[1,5]}\left(\frac{7^2 n + 1}{6}\right) q^n \equiv (\eta(6z)\eta(18z))^5 H(6z) \in \mathcal{A}_{3,5,36,1_3} \pmod{7^2}.$$

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- Let $\{g_i : 1 \leqslant i \leqslant 13\}$ be the $\mathbb{Z}$-basis for the $\mathbb{Z}$-module, $\mathcal{A}_{3,5,36,1_3} \cap \mathbb{Z}[[q]]$. Consider $\vec{g} = \langle g_i : 1 \leqslant i \leqslant 13\rangle$.

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- Basis for $M_6(\Gamma_0(3)) : \left\{ f_0(z) = \frac{\eta(z)^{18}}{\eta(3z)^6},\ f_1(z) = \eta(z)^6\ \eta(3z)^6,\ f_2(z) = \frac{\eta(3z)^{18}}{\eta(z)^6} \right\}$.
- B. - S. $\implies \exists\ H(z) \in \mathrm{M}_{36}(\Gamma_0(3))$,

$$\sum_{n=0}^{\infty} p_{[1,5]}\left(\frac{7^2 n + 1}{6}\right) q^n \equiv (\eta(6z)\eta(18z))^5 H(6z) \in \mathcal{A}_{3,5,36,1_3} \pmod{7^2}.$$

- $\mathcal{A}_{3,5,36,1_3} \cong M_{36}(\Gamma_0(3)))$ has dimension 13 whereas $\mathcal{A}_{3,5,36,1_3} \subseteq M_{41}\left(\Gamma_0(108), \left(\frac{-3}{\cdot}\right)\right)$ of dimension 729.
- Let $\{g_i : 1 \leqslant i \leqslant 13\}$ be the $\mathbb{Z}$-basis for the $\mathbb{Z}$-module, $\mathcal{A}_{3,5,36,1_3} \cap \mathbb{Z}[[q]]$. Consider $\vec{g} = \langle g_i : 1 \leqslant i \leqslant 13 \rangle$.
- With $h(z) = (\eta(6z)\eta(18z))^5 \in M_5\left(\Gamma_0(108)\left(\frac{-3}{\cdot}\right)\right)$, the basis elements are explicitly given by:

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Example 2 (Continued)

$$M \equiv \begin{pmatrix} 1 & 3 & 0 & 10 & 11 & 44 & 18 & 10 & 28 & 45 & 16 & 26 & 34 \\ 15 & 22 & 9 & 9 & 36 & 38 & 5 & 40 & 24 & 42 & 36 & 1 & 39 \\ 14 & 2 & 38 & 10 & 7 & 6 & 41 & 47 & 7 & 15 & 8 & 12 & 31 \\ 36 & 15 & 31 & 20 & 4 & 33 & 34 & 41 & 0 & 40 & 3 & 14 & 40 \\ 33 & 30 & 22 & 21 & 28 & 14 & 38 & 44 & 2 & 29 & 30 & 6 & 39 \\ 47 & 2 & 19 & 2 & 34 & 16 & 10 & 24 & 10 & 15 & 7 & 1 & 41 \\ 3 & 4 & 15 & 2 & 37 & 7 & 3 & 7 & 16 & 44 & 43 & 46 & 31 \\ 41 & 8 & 7 & 36 & 45 & 31 & 38 & 16 & 27 & 30 & 26 & 9 & 26 \\ 11 & 6 & 44 & 43 & 44 & 2 & 45 & 14 & 28 & 21 & 36 & 23 & 26 \\ 19 & 14 & 3 & 26 & 0 & 20 & 6 & 12 & 25 & 20 & 3 & 29 & 8 \\ 17 & 40 & 1 & 15 & 7 & 19 & 20 & 27 & 7 & 38 & 38 & 16 & 14 \\ 25 & 29 & 22 & 42 & 24 & 26 & 26 & 3 & 15 & 30 & 44 & 22 & 22 \\ 20 & 33 & 23 & 3 & 28 & 10 & 46 & 23 & 46 & 45 & 0 & 31 & 1 \end{pmatrix} \pmod{7^2}$$

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- The matrix

$$A = \begin{pmatrix} M - \left(\left(\frac{-3}{23}\right) 23^{40}\right) I_{13} & \left(-23^{80}\right) I_{13} \\ I_{13} & 0_{13} \end{pmatrix}$$

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- For $a \in \{0, \infty\}$, $ord_a(f_{\ell,j}(z) f_{\ell,j}(pz) \mid U_{\ell^j}) \geqslant \frac{\beta+\delta}{\ell^j}$.

Proof(Continued)

Lemma (Boylan, S.)

Consider $\lambda = \ell^j - 1 + \ell^{j-1}(\ell - 1)$, *there exists a modular form* $G(z) \in M_\lambda(\Gamma_0(p)) \cap \mathbb{Z}[[q]]$ *such that*

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$$G(z) := \ell^{\frac{\lambda}{2}-1} \underbrace{Tr_p^{\ell p}\left(\underbrace{\left(\underbrace{(f_{\ell,j}(z) f_{\ell,j}(pz)) \cdot ((h_{\ell,j}(z) h_{\ell,j}(pz)) \mid V_{\ell^{j-1}})}_{M_\lambda(\Gamma_0(\ell^j p))} \right) \mid U_{\ell^{j-1}}}_{M_\lambda(\Gamma_0(\ell p))} \mid W_\ell^{p\ell} \right)}_{M_\lambda(\Gamma_0(p))}$$

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For all $i \geqslant 1$, there exists $M_i, N_i, O_i \in Mat_{d\times d}(\mathbb{Z})$ such that

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The integers J and N arise as order of A in $PGL_{2d}(\mathbb{Z}/\ell^j\mathbb{Z})$ and $GL_{2d}(\mathbb{Z}/\ell^j\mathbb{Z})$, respectively.

Thank you

Questions?