The Weil bound for generalized Kloosterman sums of half-integral weight

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Automorphic Forms Workshop

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The Kloosterman Sum

- For $m, n, c \in \mathbb{Z}$, define

$$S(m,n,c) = \sum_{d(c)^\times} e\left(\frac{m\bar{d} + nd}{c}\right),$$

where $\bar{d}d \equiv 1 \pmod{c}$ and $e(z) = e^{2\pi i z}$.

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where $\bar{d}d \equiv 1 \pmod{c}$ and $e(z) = e^{2\pi i z}$.

- This is a *Kloosterman sum*, introduced in 1926.

The Weil Bound

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- By evaluating on prime powers and using the above bound, we have for any $c > 0$,

$$|S(m, n, c)| \leq \tau(c)(m, n, c)^{\frac{1}{2}} c^{\frac{1}{2}},$$

where τ is the divisor function.

Motivation for Generalization

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- For all $\gamma \in \Gamma_0(4)$, θ satisfies

$$\theta\left(\frac{az+b}{4cz+d}\right) = \nu_\theta(\left(\begin{smallmatrix} a & b \\ 4c & d \end{smallmatrix}\right))(4cz+d)^{\frac{1}{2}}\theta(z).$$

Motivation for Generalization, cont.

- Kohnen (1985) proved an identity for the sum

$$S(m,n,4c,\nu_\theta) := \sum_{\gamma=\left(\begin{smallmatrix} a & b \\ 4c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(4)/\Gamma_\infty} \overline{\nu}_\theta(\gamma) e\left(\frac{ma+nd}{4c}\right),$$

where $\Gamma_\infty = \langle \pm \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$.

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- The sum can be considered as over $d(4c)^\times$ and with $a \equiv \overline{d} \pmod{4c}$.
- A consequence of Kohnen's identity is that for all $c > 0$ and for all $m, n \equiv 0, 1 \pmod 4$, m a fundamental discriminant,

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- f is holomorphic on $\mathbb{H}$ and is holomorphic at $i\infty$,
- $f\left(\frac{az+b}{cz+d}\right) = \nu_f(\gamma)\,(cz+d)^k f(z)$ for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ where $\nu_f : \Gamma \to \mathbb{C}$ outputs N^{th} roots of unity for some $N \in \mathbb{N}$.

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- The function ν_f is called the *multiplier system*.

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- Let f be a modular form with multiplier system ν_f and suppose that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$.

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- Define

$$S(m, n, c, \nu_f) = \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} \overline{\nu}_f(\gamma)\, e\left(\frac{ma + nd}{c}\right),$$

where $m, n \in \mathbb{Q}$ must satisfy a consistency condition involving $\nu_f\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)$ for the sum to be well-defined.

Known Weil Bounds for Modular Kloosterman Sums

- We already know that $S(m, n, 4c, \nu_\theta) \ll_\varepsilon c^{\frac{1}{2}+\varepsilon}$.

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- Andersen (2014) proved that for $m, n \equiv 1 \pmod{24}$, m a fundamental discriminant,

$$S\left(\frac{m}{24}, \frac{n}{24}, c, \nu_\eta\right) \ll_\varepsilon c^{\frac{1}{2}+\varepsilon},$$

where η denotes the Dedekind eta function $\eta(z) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$.

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- Andersen proves an identity that is analogous to Kohnen's identity.

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- For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, there are two holomorphic choices for the square root of $cz + d$.
- The *Metaplectic group* is

$$\mathrm{Mp}_2(\mathbb{R}) = \{(M, \phi(z)) : M \in \mathrm{SL}_2(\mathbb{R}) \text{ and } \phi(z)^2 = cz + d\}$$

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  where ϕ is holomorphic on $\mathbb{H}$.
- $\mathrm{Mp}_2(\mathbb{Z})$ is the preimage of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. Also

$$\mathrm{Mp}_2(\mathbb{Z}) = \langle T, S \rangle = \left\langle \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right), \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{z}\right) \right\rangle.$$

The Weil Representation, cont.

- Let L be an even, non-degenerate lattice with symmetric bilinear form $\langle \cdot, \cdot \rangle$ and quadratic form $q(x) = \frac{1}{2} \langle x, x \rangle$ that takes its values in $\mathbb{Z}$.

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- L'/L is a finite abelian group.

The Weil Representation, cont.

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The *Weil representation* ρ_L is defined by

$$\rho_L(T)\mathfrak{e}_\alpha = e(q(\alpha))\mathfrak{e}_\alpha$$

$$\rho_L(S)\mathfrak{e}_\alpha = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|L'/L|}} \sum_{\beta \in L'/L} e(-\langle \alpha, \beta \rangle)\mathfrak{e}_\beta$$

where (b^+, b^-) is the signature of L.

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- For $\alpha, \beta \in L'/L$, let $\rho_{\alpha\beta}$ be the α, β entry in the "matrix" of ρ_L.

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$$S_{\alpha,\beta}(m,n,c) = e^{-\pi i k/2} \sum_{d(c)^\times} \overline{\rho}_{\alpha\beta}(\tilde{\gamma}) e\left(\frac{ma+nd}{2\Delta c}\right)$$

where $\tilde{\gamma} = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{cz+d}\right) \in \mathrm{Mp}_2(\mathbb{Z})$.

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- One can show

$$F(z) = \sum_{h(12)} \left(\frac{12}{h}\right) \eta(z) \mathfrak{e}_{\frac{h}{12}}$$

satisfies

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- η gives rise to a *vector-valued modular form*.

The Weil Bound

Theorem (Andersen-Anderson-W (2023))

Suppose that $g = \operatorname{rank} L$ is odd. Let $\alpha, \beta \in L'/L$ and let m, n be integers satisfying $\frac{m}{2\Delta} \in \mathbb{Z} + q(\alpha)$ and $\frac{n}{2\Delta} \in \mathbb{Z} + q(\beta)$. Write $m = m_0 v^2$ where $(-1)^{(g-1)/2} m_0$ is a fundamental discriminant. If $(v, \Delta) = 1$ then

$$S_{\alpha,\beta}(m, n, c) \ll_L 2^{\omega(c)} \tau((v, c))(m_0 n, c)^{\frac{1}{2}} c^{\frac{1}{2}},$$

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- The implied constant is of the form $|\Delta|^{Ag}$ for some absolute constant A (which is explicitly computable).

Proof Sketch

- For any $v \in \mathbb{Z}$ and $c \geq 1$, we prove the identity

$$\sum_{u|(v,c)} \left(\frac{(-4)^{(g-1)/2}m}{u}\right)\sqrt{\frac{u}{c}} S_{\alpha\frac{v}{u},\beta}\left(m\frac{v^2}{u^2}, n, \frac{c}{u}\right) = \frac{i^{-k-\frac{1}{2}(b^- - b^+)}}{\sqrt{|\Delta|}} \sum_{\substack{\ell\,(\Delta c) \\ \frac{\ell}{\Delta} \equiv \langle \alpha,\beta\rangle\,(1)}} \xi_{\alpha,\beta}(\ell,m,n,c)\, e\!\left(\frac{\ell v}{\Delta c}\right),$$

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- Möbius inversion in two variables gives

$$S_{\alpha,\beta}(m,n,c) = (\text{constant}) \times \sqrt{c} \times (\text{sparse sum}).$$

Thank You!