# Weighted Low-lying Zeros of L-functions Attached to Siegel Modular Forms

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  where $c_f$ is a parameter comparable to the analytic conductor of $L(s, f)$.
- Since $\Phi(x)$ is of rapid decay, $D(f; \Phi)$ should be regraded as capturing "low-lying" zeros of $L(s, f)$.

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- Based on Random Matrix Model prediction, we expect the following:

$$\lim_{Q\to\infty} \frac{1}{|\mathcal{F}_Q|} \sum_{f\in\mathcal{F}_Q} D(f;\Phi) = \int_{-\infty}^{\infty} \Phi(x) W(G(\mathcal{F}))(x)dx,$$

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- Here $G(\mathcal{F})$ is one of the five classic compact groups: $O$, $SO$(even), $SO$(odd), $Sp$, $U$, and $W(G(\mathcal{F}))$ is the distribution determined by low-lying eigenvalues of large random matrices of type $G(\mathcal{F})$.

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- $G(\mathcal{F})$ is called the symmetry type of $\mathcal{F}$.
- In reality, supp($\hat{\Phi}$) must be restricted to certain range. We want this range to be as large as possible.

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- Hasse-Weil $L$-functions of elliptic curves: **O**. (Young, 2006)
- Spinor (standard) $L$-functions of genus 2 Siegel modular forms: **O** (**Sp**). (Kim, Wakatsuki and Yamauchi, 2020)

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- A Siegel modular form of even weight $k \geq 6$ for $\Gamma_2$ is a holomorphic function $F : \mathcal{H}_2 \to \mathbb{C}$ such that

$$F((AZ + B)(CZ + D)^{-1}) = \det(CZ + D)^k F(Z)$$

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- Denote by $S_k(\Gamma_2)$ the space of Siegel cusp forms of weight $k$ for $\Gamma_2$.
- Each $F \in S_k(\Gamma_2)$ admits a Fourier expansion

$$F(Z) = \sum_{T \in \mathcal{T}} a_F(T)(det T)^{\frac{k}{2}-\frac{3}{4}} e(\mathrm{Tr}(TZ)),$$

  where $\mathcal{T} = \{T = (t_{ij}) \in M_2(\mathbb{R}) : T > 0, t_{11}, t_{22} \in \mathbb{Z}, 2t_{12} = 2t_{21} \in \mathbb{Z}\}$.

# Spinor and Standard L-functions

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- The spinor $L$-function $L(s, F; \mathrm{spin})$ is a degree 4 $L$-function with local parameters at $p$: $\alpha_p, \beta_p, \alpha_p^{-1}, \beta_p^{-1}$. Its functional equation was proved by Andrianov (1974):

$$\Lambda(s, F; \mathrm{spin}) = \Gamma_{\mathbb{C}}(s + 1/2)\Gamma_{\mathbb{C}}(s + k - 3/2)L(s, F; \mathrm{spin}) = \Lambda(1 - s, F; \mathrm{spin}).$$

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- The standard $L$-function $L(s, F; \mathrm{std})$ is of degree 5 with local parameters $1, \alpha_p\beta_p, \alpha_p\beta_p^{-1}, \alpha_p^{-1}\beta_p, \alpha_p^{-1}\beta_p^{-1}$. The functional equation was proved by Böcherer (1985):

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- When $F_f$ is a Saito-Kurokawa lift from $f \in S_{2k-2}(SL_2(\mathbb{Z}))$, we have

$$L(s, F_f; \mathrm{spin}) = \zeta(s + 1/2)\zeta(s - 1/2)L(s, f).$$

# Main Results

- Recall that we choose an even Schwartz function $\Phi(x)$. We also choose $c_{F;\text{spin}} = k^2$ and $c_{F;\text{std}} = k^4$, comparable to analytic conductors of $L(s, F; \text{spin})$ and $L(s, F; \text{std})$ resp.

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- With these choices, 1-level density functions $D(F; \Phi; \mathrm{spin})$ and $D(F; \Phi; \mathrm{std})$ are defined.
- For $F \in H_k(\Gamma_2)$ define harmonic weight

$$\omega_F = \frac{\sqrt{\pi}}{4}(4\pi)^{3-2k}\Gamma(k-3/2)\Gamma(k-2)\frac{a_F(I_2)^2}{||F||^2}.$$

These weights satisfy

$$\sum_{F \in H_k(\Gamma_2)} \omega_F = 1 + O(e^{-k}).$$

# Main Results

**Theorem (Zhao, 2023)**

*Suppose* $supp(\hat{\Phi}) \subset (-1,1)$. *Assume GRH for* $L(s,F;spin)$. *Then*

$$\lim_{k\to\infty} \sum_{F\in H_k(\Gamma_2)} \omega_F D(F;\Phi;spin) = \hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty} \Phi(x) W(Sp)(x)dx.$$

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- Kowalski, Saha and Tsimerman (2012) obtained the same result for $\mathrm{supp}(\hat{\Phi}) \subset (-4/15.4/15)$. Here the extension of support is crucial in application to non-vanishing problems.
- Kim, Wakatsuki and Yamauchi (2020) proved the unweighted 1-level density has orthogonal symmetry type.

# Main Results

- By extracting terms with $\rho_{F;\mathrm{spin}} = 1/2$ from

$$D(F;\Phi;\mathrm{spin}) = \sum_{\rho_{F;\mathrm{spin}}} \Phi\left(\frac{\gamma_{F;\mathrm{spin}}}{2\pi}\log k^2\right)$$

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## Corollary (Zhao, 2023)

*Assume GRH for $L(s,F;spin)$. Then we have*

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- Blomer (2019) used moments method to establish unconditionally

$$\sum_{\substack{F\in H_k(\Gamma_2)\\ L(1/2,F;\text{spin})\neq 0}} \omega_F \gg \frac{1}{\log k}.$$

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**Theorem (Zhao, 2023)**

*Suppose* $supp(\hat{\Phi}) \subset (-1/4, 1/4)$. *Assume GRH for* $L(s, F; std)$. *Then*

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- Kim, Wakatsuki and Yamauchi (2020) found the corresponding unweighted 1-level density has symplectic symmetry type.
- Previously it was not known what the weighted 1-level density symmetry type is. Our result indicates that it remains symplectic.

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*Suppose* $supp(\hat{\Phi}) \subset (-5/18, 5/18)$. *Assume GRH for* $L(s, F; std)$. *Then*

$$\lim_{K\to\infty} \frac{2}{K} \sum_{k\sim K} \sum_{F\in H_k(\Gamma_2)} \omega_F D(F;\Phi;std) = \hat{\Phi}(0) - \frac{\Phi(0)}{2} = \int_{-\infty}^{\infty} \Phi(x) W(Sp)(x) dx.$$

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- Unfortunately, this range of support $(-5/18, 5/18)$ is still not large enough to say anything about non-vanishing of $L(1/2, F; \mathrm{std})$.

# Proof Ingredients

- Explicit formula

$$D(F;\Phi;\mathrm{spin}) = \frac{2}{2\pi i}\int_{(2)} \Phi\left(\frac{s-\frac{1}{2}}{2\pi i}\log k^2\right)\frac{\Lambda'}{\Lambda}(s,F;\mathrm{spin})ds + (\cdots).$$

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- Apply a Petersson-type trace formula (Kitaoka, 1984).
- For averaging over $k$, apply a integral representation of products of Bessel functions. Then apply the method of Neumann series.