Arithmeticity of automorphic forms

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Outline

Example of arithmeticity

Suppose $\ell \geq 0$ is an integer, $N \geq 1$ is an integer, and $f : \mathfrak{h} \to \mathbf{C}$ is a holomorphic modular form of weight ℓ and level $\Gamma_1(N)$.

Fourier expansion

If $z \in \mathfrak{h}$, then $f(z) = \sum_{m \geq 0} a(m) q^m$, where $q = e^{2\pi i z}$, $a(m) \in \mathbf{C}$ are the Fourier coefficients.

Modular forms can be **arithmetic**. For example:

1. Let $\theta(z) = \sum_{n \in \mathbf{Z}} q^{n^2}$, $f(z) = \theta(z)^k$ for some $k \geq 1$. Then

$$f(z) = \sum_{m \geq 0} r_k(m) q^m$$

where $r_k(m)$ is the number of ways of writing m as the sum of k-squares:

$$r_k(m) = \#\{(x_1, \ldots, x_k) \in \mathbf{Z}^k : x_1^2 + \cdots + x_k^2 = m\}.$$

Measuring arithmeticity

2. More generally, instead of the Fourier coefficients $a(m)$ being the solution to a **counting problem**, one could ask $a(m) \in \mathbf{Z}$ for all m. For example:

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots$$

3. One could instead **evaluate** a modular form at a point: if f is a modular form of weight 4ℓ, $\ell \geq 0$ an integer, then can ask:

$$\frac{f(\tau)}{E_4(\tau)^\ell} \stackrel{?}{\in} \overline{\mathbf{Q}}$$

for certain τ, e.g., $\tau = \sqrt{-1}$.

Arithmeticity via Fourier coefficients

Definition

Suppose $R \subseteq \mathbf{C}$ is a subset, $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ is a congruence subgroup and $\ell \geq 0$ is an integer.

$$M_\ell(\Gamma, R) := \{f \in M_\ell(\Gamma) : a_f(m) \in R \text{ for all } m\}.$$

One says that a modular form f is **arithmetic** if $f \in M_\ell(\Gamma, \overline{\mathbf{Q}})$.

Classic result:

Theorem (Shimura)

One has

$$M_\ell(\Gamma, \mathbf{C}) = M_\ell(\Gamma, \mathbf{Q}^{cyc}) \otimes_{\mathbf{Q}^{cyc}} \mathbf{C}.$$

In other words, the modular forms of weight ℓ and level Γ have a basis $f_1, \ldots, f_L$, so that $a_{f_j}(m) \in \mathbf{Q}^{cyc}$ for all j, m.

Arithmeticity via evaluation

Definition

A point $\tau \in \mathfrak{h}$ is said to be CM if $\tau = a + b\sqrt{-D}$ some $a, b \in \mathbf{Q}$, $D \in \mathbf{Z}_{\geq 1}$.

Definition

A meromorphic function $f = \frac{g}{h}$ with $g, h \in M_\ell(\Gamma)$ is said to be **arithmetic** if $f(\tau) \in \overline{\mathbf{Q}}$ whenever τ is CM and $h(\tau) \neq 0$.

Theorem (Shimura)

If $g, h \in M_\ell(\Gamma, \overline{\mathbf{Q}})$, τ is CM and $h(\tau) \neq 0$, then $\frac{g(\tau)}{h(\tau)} \in \overline{\mathbf{Q}}$. That is, gh^{-1} is **arithmetic**. *In fact, given a CM τ, there exists $u_\tau \in \mathbf{C}^\times$ so that if $g \in M_\ell(\Gamma, \overline{\mathbf{Q}})$ then $u_\tau^{-\ell} g(\tau) \in \overline{\mathbf{Q}}$.*

Modular curve

Suppose $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ is a congruence subgroup.

Modular curve

Set

$$X(\Gamma) = \Gamma\backslash\mathfrak{h} \ \sqcup \ \Gamma\backslash\mathbf{P}^1(\mathbf{Q}).$$

One can give $X(\Gamma)$ the structure of a connected, compact manifold of dimension 2.

Application: Relate arithmetic geometry and modular forms

Fix a number field $E_\Gamma \subseteq \mathbf{Q}^{cyc} \subseteq \mathbf{C}$. Set $R_\Gamma = \bigoplus_{\ell \geq 0} M_\ell(\Gamma, E_\Gamma)$, so that R_Γ is a graded ring. Then $X_\Gamma := \mathrm{Proj}(R_\Gamma)$ is a projective variety defined over E_Γ.

1. One has $X_\Gamma(\mathbf{C}) = X(\Gamma)$
2. If τ is CM, the *evaluation* theorem above implies that the image of τ in $X(\Gamma)$ comes from a point in $X_\Gamma(\overline{\mathbf{Q}})$.

Arithmeticity via singular cohomology

Eichler-Shimura isomorphism

The map

$$S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \to H^1(X(\Gamma), \mathbf{C})$$

given by

$$(f, \overline{g}) \mapsto f(z)(dx + idy) + \overline{g(z)}(dx - idy)$$

induces an isomorphism into cohomology.

Rational structure

Topology gives the complex vector space $H^1(X(\Gamma), \mathbf{C})$ a rational structure:

$$H^1(X(\Gamma), \mathbf{C}) = H^1(X(\Gamma), \mathbf{Q}) \otimes \mathbf{C}.$$

This rational structure is **not** the same rational structure on $S_2(\Gamma)$ given by Fourier coefficients.

Arithmeticity via Fourier coefficients: Proof

Beginning of proof

1. Calculate explicitly the Fourier coefficients of Eisenstein series.
2. Thus, reduced to cuspidal case.
3. Fix $E \in M_r(\Gamma, \mathbf{Q}^{cyc})$, $r > 0$, $E \neq 0$. By multiplying by E, obtain injection
$$S_\ell(\Gamma) \to S_{\ell+r}(\Gamma).$$
Can use this to reduce to the case of $\ell >> 0$.

Arithmeticity of cusp forms in sufficiently large weight

1. **Way 1**: If $f \in S_\ell(\Gamma)$ is an eigenform, can prove $\langle f, E_t E_{\ell-t}\rangle \neq 0$ using Rankin-Selberg. Thus, the Hecke submodule generated by products of two Eisenstein series contains all of $M_\ell(\Gamma)$.
2. **Way 2**: Again using Rankin-Selberg, can prove every cuspidal f is a harmonic theta function

Arithmeticity via evaluation: proof

Proof sketch

1. Suppose τ is a CM point. Then $E_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ has ring of endomorphisms R equal to an order in $\mathbf{Q}(\tau)$.
2. Using the Weierstrass $\wp$-function, E_τ is the $\mathbf{C}$-points of an elliptic curve $y^2 = x^3 + ax + b$.
3. The group $\mathrm{Aut}(\mathbf{C})$ acts on the (finite!) set of isomorphism classes of elliptic curves with endormorphism ring equal to R.
4. Let $j(\tau) = q^{-1} + 744 + 196884q + \cdots$ be the j-function. It follows that $j(\tau)$ lives in a number field, and that there exists $u_\tau \in \mathbf{C}^\times$ so that $\Delta(\tau)/u_\tau^{12} \in \overline{\mathbf{Q}}$.
5. Suppose $f \in M_\ell(\mathrm{SL}_2(\mathbf{Z}))$. Then f^{12}/Δ^ℓ is a polynomial in the j-function. If $a_f(m) \in \overline{\mathbf{Q}}$, this polynomial has coefficients in $\overline{\mathbf{Q}}$.
6. For deeper level Γ: For $\delta \in \Gamma\backslash\mathrm{SL}_2(\mathbf{Z})$, set $f_\delta = f|_\ell\delta = j(\delta, z)^{-\ell} f(\delta z)$. The polynomial $P(X) = \prod_\delta (X - f_\delta)$ has f as a root, and coefficients of level one.

Other applications

Another way to obtain algebraicity of Hecke eigenvalues

Suppose $\ell \geq 2$ and $f \in S_\ell(\Gamma)$ is a Hecke eigenform: $T(n)f = \lambda(n)f$. Then $\lambda(n) \in \overline{\mathbf{Q}}$ for all n.

Proof: The Hecke operators preserve the rational structure $H^1(X(\Gamma), \mathbf{Q})$, and thus their eigenvalues are algebraic.

Applications to L-values

1. *L*-**values**: Can express certain L-values $L(f, \chi, s_0)$ in terms of the pairing between cohomology and homology (Hecke integral). Using this, can prove $\frac{L(f,\chi,1/2)}{L(f,1/2)} \in \mathbf{Q}_f(\chi)$ when $L(f, 1/2) \neq 0$.
2. **More** *L*-**values**: Using Eisenstein series and Fourier coefficients, can prove statements of the form $\frac{L(f \times g, s_0)}{\langle f, f \rangle} \in \mathbf{Q}_f \cdot \mathbf{Q}_g$.

Outline

Siegel modular forms

Symplectic group

Let $n \geq 1$ be an integer and set $J_n = \left(\begin{smallmatrix} 0_n & 1_n \\ -1_n & 0_n \end{smallmatrix}\right)$. One sets

$$\mathrm{Sp}_{2n} := \{g \in \mathrm{GL}_{2n} : g^t J_n g = J_n\}.$$

Siegel upper half space

Let

$$\mathcal{H}_n = \{Z = X + iY \in M_n(\mathbf{C}) : Z^t = Z, Y > 0\}.$$

If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbf{R})$ and $Z \in \mathcal{H}_n$, set

$$g \cdot Z = (aZ + b)(cZ + d)^{-1}.$$

This defines an action $\mathrm{Sp}_{2n}(\mathbf{R}) \times \mathcal{H}_n \to \mathcal{H}_n$.

Set $j(g, Z) = \det(cZ + d)$.

Fourier expansion

Siegel modular forms

Fix $\ell \in \mathbf{Z}_{\geq 0}$ and $\Gamma \subseteq \mathrm{Sp}_{2n}(\mathbf{R})$ arithmetic subgroup. A function $f : \mathcal{H}_n \to \mathbf{C}$ is said to be a **Siegel modular form** of weight ℓ and level Γ is f is holomorphic, of moderate growth, and $f(\gamma Z) = j(\gamma, Z)^{\ell} f(Z)$ for all $\gamma \in \Gamma$.

Suppose for simplicity that $\Gamma = \mathrm{Sp}_{2n}(\mathbf{Z})$.

Fourier coefficients

If f is a Siegel modular form, then

$$f(Z) = \sum_{T \geq 0} a_f(T) e^{2\pi i \,\mathrm{tr}(TZ)},$$

for some Fourier coefficients $a_f(T) \in \mathbf{C}$. The sum ranges over $T \in \frac{1}{2} M_n(\mathbf{Z})$ with T symmetric, i.e., $T^t = T$.

Arithmeticity for Siegel modular forms

Definition

Suppose $\Gamma \subseteq \mathrm{Sp}_{2n}(\mathbf{Z})$ is an arithmetic subgroup and $R \subseteq \mathbf{C}$ is a subset. Let $M_\ell(\Gamma, R)$ denote the Siegel modular forms f of weight ℓ, all of whose Fourier coefficients $a_f(T)$ live in R.

Theorem (Shimura)

The Fourier coefficients give Siegel modular forms an arithmetic structure, i.e., one has

$$M_\ell(\Gamma, \mathbf{C}) = M_\ell(\Gamma, \mathbf{Q}^{cyc}) \otimes_{\mathbf{Q}^{cyc}} \mathbf{C}.$$

Obtain connection[1] with arithmetic geometry.

[1] Actually, Shimura *first* proved the existence of the canonical model, and used that to prove the arithmetic structure via Fourier coefficients.

Ideas that do and don't quite work

These ideas don't work

1. Using singular cohomology of $\Gamma\backslash\mathcal{H}_n$, can prove that the Hecke eigenvalues of Siegel modular forms are algebraic.
2. However, if $n \geq 2$, the Fourier coefficients of Siegel modular forms contain much more information than the Hecke eigenvalues, so that approach doesn't work.

These ideas kinda work

1. One can use theta functions to prove arithmeticity of Siegel modular forms, but that doesn't work to prove arithmeticity on other groups that have holomorphic modular forms (e.g., $SO(2, n)$)
2. Likewise, using Eisenstein series and Rankin-Selberg integrals (Garrett), one can prove arithmeticity of Siegel modular forms, but that won't generalize to other groups.

Motivating question

Principle

Certain classes of (non-holomorphic) automorphic forms **should** (also) be arithmetic.

Singular cohomology

Sometimes, if $\varphi : \Gamma \backslash G \to \mathbf{C}$ is an automorphic form, then φ defines a (deRham) cohomology class $c(\varphi) \in H^j(X(G;\Gamma), \mathbf{C})$,

$$X(G;\Gamma) := \Gamma \backslash G / K,$$

K a maximal compact subgroup of G. Then one can say φ is *topologically* arithmetic if $c(\varphi) \in H^j(X(G;\Gamma), \mathbf{Q})$.

- Singular cohomology misses a lot of very useful structure.

Measuring arithmeticity?

In general, how can one even **measure** or **define** notions of arithmeticity, let alone prove an arithmetic structure exists?

Outline

Fourier and Fourier-Jacobi coefficients

Suppose f is a Siegel modular form with Fourier coefficients $a_f(T)$.

Basic fact

If $\gamma \in \mathrm{SL}_n(\mathbf{Z})$, then $a_f(\gamma^t T \gamma) = a_f(T)$.

Let $n = 2$ for simplicity. Fix integers $m \geq 1$ and μ.

Definition

Suppose f is a Siegel modular form. Define $h_{m,\mu}(D) = a_f(T)$ if $T = \begin{pmatrix} m & \mu/2 \\ \mu/2 & * \end{pmatrix}$ and $\det(T) = D$.

Proposition

Notation as above. Then

$$\sum_{D \geq 0} h_{m,\mu}(D) q^D \in M_{\ell-1/2}(\Gamma_0(4m))$$

for every congruence class μ modulo $2m$.

Formal modular forms

Names in alphabetical order, some in collaboration with others, and some with multiplicity > 1:

Definition: Aoki, Bruinier, Ibukiyama, Pollack, Poor, Raum, Yuen

A function $A : 2^{-1}M_2(\mathbf{Z}) \to \mathbf{C}$ is said be a **formal modular form** if it satisfies the following conditions:

1. $A(T) \neq 0$ implies $T \geq 0$;
2. $A(\gamma^t T \gamma) = a(T)$ for all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$;
3. For every m, μ, define $h_{m,\mu}(D)$ as above in terms of $A(T)$. Then
$$\sum_{D \geq 0} h_{m,\mu}(D) q^D$$
is the Fourier expansion of a modular form in $M_{\ell-1/2}(\Gamma_0(4m))$.

Sending a Siegel modular form to its Fourier coefficients defines an injection
$$M_\ell(\Gamma) \to M_\ell^f(\Gamma)$$

Automatic convergence

Authors on previous slide, in differing degrees of generality:

Theorem

The injection $M_\ell(\Gamma) \to M_\ell^f(\Gamma)$ is an isomorphism. In particular, if A is a formal modular form, then $|A(T)|$ grows polynomially with T.

Convergence is the whole point

Suppose A is a formal modular form. **Once one knows** that the $A(T)$ grow slowly, then one can define $f(Z) = \sum_T A(T)e^{2\pi i\,\mathrm{tr}(TZ)}$. The sum converges to a holomorphic function on $\mathcal{H}_2$. The conditions satisfied by A imply that f is a Siegel modular form.

Corollary (New proof (P.) of results of Shimura, Harris)

For Γ an arithmetic subgroup of Sp_{2n}, $\mathrm{SO}(2,n)$, $\mathrm{U}(n,n)$, $\mathrm{SO}^*(4n)$, or $E_{7,3}$, one has

$$M_\ell(\Gamma, \mathbf{C}) = M_\ell(\Gamma, \mathbf{Q}^{cyc}) \otimes_{\mathbf{Q}^{cyc}} \mathbf{C}.$$

Proof sketch

Proof: (Argument of P).

1. Suppose A is a cuspidal formal modular form, i.e., $A(T) \neq 0$ implies $T > 0$.
2. **Induct** on floor of $\log(\log(\det(T)))$.
3. Suppose $\ell^2 \cdot 2^{2^r} \leq \det(T) \leq \ell^2 \cdot 2^{2^{r+1}}$, and one has bound on $|A(S)|$ for all S with $\det(S) \leq \ell^2 \cdot 2^{2^r}$.
4. **Reduction theory**: There exists $\gamma \in SL_2(\mathbf{Z})$ so that if $T' = \gamma^t T \gamma$, then $m' := m(T') \leq \det(T)^{1/2} \leq \ell \cdot 2^{2^r}$.
5. The number $A(T) = A(T')$ appears as a Fourier coefficient of $f_{m'} \in M_{\ell-1/2}(\Gamma_0(4m'))$.
6. By a **Sturm bound**, the modular form $f_{m'}$ is determined by the Fourier coefficients $h_{m',\mu}(D)$ with $D \leq \ell \cdot m' \leq \ell^2 \cdot 2^{2^r}$.
7. The $h_{m',\mu}(D)$'s are $A(S)$ for some S. One can **propagate** the bound on the $A(S)$ to a bound on $A(T)$.

□

Let $G \in \{SO(4,n), G_2, F_{4,4}, E_{6,4}, E_{7,4}, E_{8,4}\}$, $K \subseteq G$ is a maximal compact subgroup. Then

$$K \twoheadrightarrow SU(2)/\mu_2 \simeq SO(3).$$

Let $\ell \geq 1$ be an integer. Set $\mathbf{V}_\ell = Sym^{2\ell}(\mathbf{C}^2)$ as a representation of $K \to SU(2)/\mu_2 = SO(3)$.

Definition (Gross-Wallach, Gan-Gross-Savin)

A **quaternionic modular form** of level Γ and weight ℓ is a smooth, moderate growth function $\varphi : G \to \mathbf{V}_\ell$ satisfying

1. $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$
2. $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $k \in K$
3. φ satisfies a particular special linear differential equation, $D_\ell \varphi \equiv 0$

Fourier coefficients of quaternionic modular forms

Fourier expansion

If φ is a quaternionic modular form of weight ℓ, then

$$\varphi(g) \text{``}=\text{''} \sum_{\lambda \in \Lambda_\Gamma} a_\varphi(\lambda) \mathcal{W}_\lambda(g).$$

Here

- $\mathcal{W}_\lambda(g) : G \to \mathbf{V}_\ell$ are completely explicit functions, independent of φ
- $a_\varphi(\lambda) \in \mathbf{C}$ are called the Fourier coefficients of φ.

If $R \subseteq \mathbf{C}$, write $S_\ell(G, \Gamma, R)$ for the cuspidal quaternionic modular forms of weight ℓ on G of level Γ, with all Fourier coefficients in R.

Theorem (P.)

Suppose $G \in \{F_{4,4}, E_{6,4}, E_{7,4}, E_{8,4}\}$ has real and rational rank equal to 4. Then $S_\ell(G, \Gamma; \mathbf{C}) = S_\ell(G, \Gamma; \overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$.

Thank you

Thank you for your attention!