Frobenius-Schur Indicators for p-adic Special Linear Groups

Ergun Suer

University of Oklahoma

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Plan

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Some definitions

Let G be a totally disconnected locally compact group. A (complex) **representation** (π, V) of G is a group homomorphism

$$\pi\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

where V is a complex vector space. A representation (π, V) of G is called **smooth** if for every $v \in V$ the stabilizer

$$\mathrm{Stab}_G(v) = \{g \in G : \pi(g)v = v\}$$

is an open subgroup of G.
A **character** of G is a one-dimensional smooth representation of G.

Some definitions

Let (π, V) be a smooth representation of G. Let $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the algebraic dual of V. The group G acts on V^* by

$$(g \cdot \lambda)(v) = \lambda(\pi(g^{-1})v), \qquad g \in G,\ \lambda \in V^*,\ v \in V.$$

An element $\lambda \in V^*$ is called *smooth* if its stabilizer

$$\mathrm{Stab}_G(\lambda) = \{g \in G : g \cdot \lambda = \lambda\}$$

is an open subgroup of G. Let $V^\vee$ be the subspace of smooth elements in V^*. The **contragredient** representation $(\pi^\vee, V^\vee)$ is the representation obtained by restricting the above action of G on V^* to $V^\vee$.

Some definitions

A smooth representation (π, V) of G is called **irreducible** if V has no nonzero proper G-stable subspaces.
For any irreducible smooth representation (π, V) of G there exists a character

$$\omega_\pi \colon Z(G) \to \mathbb{C}^\times$$

such that $\pi(z)$ acts as the scalar $\omega_\pi(z)$ on V for all $z \in Z(G)$. We call ω_π the **central character** of π.

History: Frobenius-Schur Indicator "sign"

G finite group

(π, V) irreducible (complex) rep of G

χ character of π $\quad (\chi = \mathrm{Tr} \circ \pi)$

We say π can be realized over $\mathbb{R}$

if $\exists$ basis of V s.t. $\forall g \in G$, the corresponding matrix $[\pi(g)]$ has all real entries

Question. When can π be realized over $\mathbb{R}$?

- realized
  - χ real valued
  - $\varepsilon(\pi) = 1$
- not realized
  - χ real
    - $\varepsilon(\pi) = -1$
  - χ not real valued
    - $\varepsilon(\pi) = 0$

History: Frobenius Schur Indicator "sign"

Frobenius and Schur (1906):
Let G be a finite group and let π be an irreducible complex representation of G with character χ_π. The Frobenius–Schur indicator is defined by

$$\epsilon(\pi) = \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g^2).$$

An equivalent formulation:

$$\epsilon(\pi) = \begin{cases} 1, & \text{if } \exists \text{ nonzero } G\text{-invariant symmetric bilinear form on } V, \\ -1, & \text{if } \exists \text{ nonzero } G\text{-invariant skew-symmetric bilinear form on } V, \\ 0, & \text{if } \pi \not\cong \pi^\vee. \end{cases}$$

History: Some Results

Gow (1983 and 1985):

- For every irreducible self-dual representation π of $G := \mathrm{GL}(n, \mathbb{F}_q)$,

$$\epsilon(\pi) = 1.$$

Let q be a power of an odd prime.

- For every irreducible self-dual representation π of $G := \mathrm{SO}(n, \mathbb{F}_q)$,

$$\epsilon(\pi) = 1.$$

- For any irreducible self-dual representation π of $G = \mathrm{Sp}(2n, \mathbb{F}_q)$,

$$\epsilon(\pi) = \omega_\pi(-I).$$

Vinroot (2006): Let F be a non-archimdean local field.

- For every irreducible self-dual representation π of $G := GL(n, F)$,

$$\epsilon(\pi) = 1.$$

Problem

F non-archimdean local field
$G = \mathrm{SL}(n, F)$
$\pi \in \mathrm{Irr}(G)$ with $\pi \simeq \pi^\vee$

What can we say about $\epsilon(\pi)$?

Strategy

Reduce the problem to *generic* representations to use *D. Prasad's idea*.

Strategy

Theorem [D. Prasad's theorem (1999)].
Let G be the group of $F-$points of a connected reductive group defined and split over F. Let B be a Borel subgroup of G with Levi decomposition $B = TU$. Assume that there exists an element $s \in T$ such that

$$\alpha(s) = -1 \qquad \text{for all simple roots } \alpha.$$

Then
(1) $s^2 \in Z(G)$
(2) If σ_0 is an irreducible self dual generic representation of G

$$\epsilon(\sigma_0) = \omega_{\sigma_0}(s^2).$$

Definition

Let $G = \mathrm{GL}(n, F)$. Fix the standard Borel subgroup $B = TU$
A character $\psi : U \to \mathbb{C}^\times$ is called **nondegenerate** if its restriction to each simple root subgroup

$$U_i = \{I + xE_{i,i+1} : x \in F\}, \qquad 1 \leq i \leq n-1,$$

is nontrivial.

Definition

Let $G = \mathrm{GL}(n, F)$, and fix the standard Borel subgroup $B = TU$. Let $\psi : U \to \mathbb{C}^\times$ be a nondegenerate smooth character. An irreducible smooth representation (π, V) of G is called ψ-generic if

$$\mathrm{Hom}_U(\pi, \psi) \neq 0.$$

The representation π is called **generic** if it is ψ-generic for some *nondegenerate* character ψ of U.

Roadmap

1. Step 1: Reduction to tempered case: Roche-Spallone (2012)
2. Step 2: Lifting and reducing process
3. Step 3: Application of twisted version of D. Prasad's theorem (1999)

Step 1: Reduction to tempered case: Twisted Signs

Let F be a non-Archimedean local field and G be the group of F-points of a connected reductive algebraic group. Let (π, V) be an irreducible representation of G. Let θ be a continuous automorphism of G of order at most two. Let (π^θ, V) be the θ-twisted of π defined by

$$\pi^\theta(g)v = \pi(\theta(g))v \text{ for all } g \in G, v \in V.$$

Suppose that $\pi^\theta \simeq \pi^\vee\chi$ for some character χ of G.

Step 1: Reduction to tempered case: Twisted Signs

Note that

$$\mathrm{Hom}(\pi^\theta, \pi^\vee \chi) \simeq \mathrm{Bil}_{\pi\chi^{-1},\pi^\theta}(V)$$

where $\mathrm{Bil}_{\pi\chi^{-1},\pi^\theta}(V)$ is the space of all bilinear forms b on V such that

$$b(\pi(g)\chi^{-1}(g)v_1, \pi^\theta(g)v_2) = b(v_1, v_2)$$

for all $g \in G, v_1, v_2 \in V$.
By Schur's lemma, any nonzero form b in $\mathrm{Bil}_{\pi\chi^{-1},\pi^\theta}(V)$ is either symmetric or skew-symmetric. Accordingly, we set

$$\epsilon_{\theta,\chi}(\pi) = \begin{cases} 1 & \text{if } b \text{ is symmetric,} \\ -1 & \text{if } b \text{ is skew-symmetric.} \end{cases}$$

If θ is the trivial automorphism of G and χ is the trivial character of G, we simply write $\epsilon(\pi)$ and call it the **ordinary sign**. Similarly, if θ is trivial, we write $\epsilon_\chi(\pi)$, and if χ is trivial, we write $\epsilon_\theta(\pi)$.

Step 1: Reduction to tempered case: Twisted Signs

Proposition

Let (π, V) be an irreducible representation of G with $\pi \simeq \pi^\vee$. Let $x \in G$ such that $x^2 \in Z(G)$. Let θ be the inner automorphism of G defined by $\theta(g) = xgx^{-1}$ for all $g \in G$. Then we have

$$\epsilon(\pi) = \epsilon_\theta(\pi)\omega_\pi(x^2)$$

where ω_π is the central character of π.

Step 1: Reduction to tempered case: Twisted Signs

Theorem [Roche-Spallone (2012)].
Let G be the group of F-points of a connected reductive algebraic group. Let θ be an involutory automorphism of G such that $\pi^\theta \simeq \pi^\vee$. Suppose that P has Levi decomposition $P = MN$. Let (P, σ, ν) be the triple associated to π via the Langlands classification. Under certain assumptions on the involution θ, they prove the following:

$$\epsilon_\theta(\pi) = \epsilon_\theta(\sigma)$$

.

Definition

Let (π, V) be an irreducible representation of $G = \mathrm{GL}(n, F)$ with *unitary* central character. We say that π is **tempered** if for all $v \in V$, $\lambda \in V^\vee$, and every $\varepsilon > 0$, the function

$$gZ(G) \longmapsto |f_{v,\lambda}(g)|$$

belongs to $L^{2+\varepsilon}(G/Z(G))$.

Step 1: Settings

F non-archimdean local field
$G = SL(n, F)$
$\pi \in \mathrm{Irr}(G)$ with $\pi \simeq \pi^\vee$

By Langlands theory, there exists a unique triple (P, σ, ν) such that π is the unique irreducible quotient of $i_P^G(\sigma\nu)$.
Note that

$$M = \begin{bmatrix} \mathrm{GL}_{n_1}(F) & & \\ & \ddots & \\ & & \mathrm{GL}_{n_r}(F) \end{bmatrix}_{n\times n} \cap G.$$

Step 1: Palindromic property of the block sizes

Uniqueness of Langlands triple (P, σ, ν) and self-duality of π implies

$$(n_1, n_2, \ldots, n_r) = (n_r, n_{r-1}, \ldots, n_1)$$

So

$$n = 2(n_1 + n_2 + \cdots + n_\ell) + n_0$$

where n_0 is the size of the middle block in M.

Step 1: Involution $\theta \in Aut(G)$

Let θ be the inner automorphism of G given by the matrix $w_\theta \in G$, where w_θ is equal to one of the three matrices:

- If $n_0 = 0$ then

$$w_\theta = \begin{bmatrix} & & & & & & -I_{n_1} \\ & & & & & -I_{n_2} & \\ & & & & \iddots & & \\ & & & -I_{n_\ell} & & & \\ & & I_{n_\ell} & & & & \\ & \iddots & & & & & \\ & I_{n_2} & & & & & \\ I_{n_1} & & & & & & \end{bmatrix}_{n \times n}$$

Step 1: Involution $\theta \in Aut(G)$

- If $n_1 + n_2 + \cdots + n_\ell$ is even and $n_0 \neq 0$ then

$$w_\theta = \begin{bmatrix} & & & & & & I_{n_1} \\ & & & & & I_{n_2} & \\ & & & & \iddots & & \\ & & & I_{n_0} & & & \\ & & \iddots & & & & \\ & I_{n_2} & & & & & \\ I_{n_1} & & & & & & \end{bmatrix}_{n \times n}$$

Step 1: Involution $\theta \in Aut(G)$

- If $n_1 + n_2 + \cdots + n_\ell$ is odd and $n_0 \neq 0$ then

$$w_\theta = \begin{bmatrix} & & & & & & I_{n_1} \\ & & & & & \iddots & \\ & & & & I_{n_\ell} & & \\ & & & \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{n_0 \times n_0} & & & \\ & & I_{n_\ell} & & & & \\ & \iddots & & & & & \\ I_{n_1} & & & & & & \end{bmatrix}_{n \times n}$$

Step 1: Reduction to tempered case

Then, we get $\epsilon(\pi) = \begin{cases} \epsilon_\theta(\pi)\omega_\pi(-1) & \text{if } n_0 = 0, \\ \epsilon_\theta(\pi) & \text{if } n_0 \neq 0 \end{cases}$

$$\epsilon_\theta(\pi) = \epsilon_\theta(\sigma)$$

Step 2: Lifting from SL-Levi M to GL-Levi $\widetilde{M}$

We transfer our problem to

$$\widetilde{M} = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_l}(F) \times \mathrm{GL}_{n_0}(F) \times \mathrm{GL}_{n_l}(F) \times \cdots \times \mathrm{GL}_{n_1}(F).$$

- There exists an irreducible tempered representation $\widetilde{\sigma}$ of $\widetilde{M}$ such that σ occurs in $\widetilde{\sigma}|_M$.
- There exists a smooth character $\chi : \widetilde{M} \to \mathbb{C}^\times$ which is trivial on M such that
$$\widetilde{\sigma}^\theta \simeq \widetilde{\sigma}^\vee \chi.$$
- We have $\epsilon_\theta(\sigma) = \epsilon_{\theta,\chi}(\widetilde{\sigma})$.

Step 2: Reducing to the middle GL_{n_0}-block

Note that

$$\widetilde{\sigma} = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_\ell \otimes \sigma_0 \otimes \sigma'_\ell \otimes \cdots \otimes \sigma'_2 \otimes \sigma'_1$$

for some irreducible representations σ_1, σ'_1 of $\mathrm{GL}_{n_1}(F), \sigma_2, \sigma'_2$ of $\mathrm{GL}_{n_2}(F), \ldots, \sigma_\ell, \sigma'_\ell$ of $\mathrm{GL}_{n_\ell}(F)$, and σ_0 of $\mathrm{GL}_{n_0}(F)$.

Step 2: Reducing to the middle GL_{n_0}-block

After substitution, we have

$$\widetilde{\sigma} \simeq \sigma_1 \otimes \sigma_2 \otimes \dots \sigma_\ell \otimes \sigma_0 \otimes \sigma_\ell^\vee \chi \otimes \cdots \otimes \sigma_2^\vee \chi \otimes \sigma_1^\vee \chi.$$

Then we have

$$\epsilon_{\theta,\chi}(\widetilde{\sigma}) = \epsilon_{\theta_0,\chi_0}(\sigma_0) = \epsilon_{\chi_0}(\sigma_0)\chi_0(\Sigma_\theta),$$

where $\theta_0 \in Inn(\mathrm{GL}_{n_0}(F))$ given by the middle block matrix Σ_θ of w_θ, and $\chi_0 = \chi|_{\mathrm{GL}_{n_0}(F)}$

Step 3: Application of twisted version of Prasad's theorem

By twisted version of Prasad's theorem, we have

$$\begin{aligned}\epsilon_{\theta,\chi}(\widetilde{\sigma}) = \epsilon_{\theta_0,\chi_0}(\sigma_0) &= \epsilon_{\chi_0}(\sigma_0)\chi_0(\Sigma_\theta)\\ &= \omega_{\sigma_0}(s^2)\chi_0^{-1}(s)\chi_0(\Sigma_\theta),\end{aligned}$$

where $s \in T_0$ acts by -1 on all the simple roots in U_0, equivalently, $s \in \{a \cdot \mathrm{diag}(1, -1, 1, -1, \ldots, (-1)^{n_0-1}) : a \in F^\times\}$.

A structural sign formula

Let π be an irreducible self-dual representation of $\mathrm{SL}(n, F)$, and let $(n_1, \ldots, n_\ell, n_0, n_\ell, \ldots, n_1)$ be the block sizes in the Levi attached to the Langlands data for π. Then

$$\epsilon(\pi) = \begin{cases} \omega_\pi(-1), & \text{if } n_0 = 0, \\ \chi(-1)^{\lfloor n_0/2 \rfloor + n_1 + \cdots + n_\ell}, & \text{if } n_0 \neq 0, \end{cases}$$

Main Theorem

Theorem. Let π be an irreducible representation of $G = \mathrm{SL}(n, F)$ with $\pi \simeq \pi^\vee$, where F is a nonarchimedean local field.

1. If n is odd, then $\epsilon(\pi) = 1$.
2. If $n \equiv 0 \pmod 4$, then $\epsilon(\pi) = 1$.
3. If $n \equiv 2 \pmod 4$ and F contains $\sqrt{-1}$, then

$$\epsilon(\pi) = \omega_\pi(-I),$$

Example of SL(6,F)

If F be a non-Archimedean local field whose residue field has cardinality $q \equiv 3 \pmod 4$ then there exists an irreducible self-dual generic representations π of $\mathrm{SL}(6, F)$ such that

$$\epsilon(\pi) = -1 \text{ even though } \omega_\pi(-I) = 1$$

Open questions

1. Determine a computable criterion for $\epsilon(\pi)$ when π is an irreducible self-dual representation of $\mathrm{SL}(n, F)$ with $n \equiv 2 \pmod 4$ and $\sqrt{-1} \notin F$.
2. Determine signs of irreducible self-dual representations for other classical groups over non-Archimedean local fields, particularly: '
   - the symplectic groups Sp(2n, F),
   - the special orthogonal groups SO(n, F),
   - the unitary groups SU(n, F).

Thank you!

References

[1] R. Gow, *Real representations of the finite orthogonal and symplectic groups of odd characteristic*, J. Algebra (1985)
[2] Dipendra Prasad, *On the self-dual representations of finite groups of Lie type*, J. Algebra (1998)
[3] Dipendra Prasad, *On the self-dual representations of a p-adic group*, Internat. Math. Res. (1999)
[4] Alan Roche and Steven Spallone, *Signs, Involutions and Jacquet Modules*, preprint (2012)
[5] Marko Tadić, *Notes on representations of non-Archimedean SL(n)*, Pacific J. Math. (1992), no 2, 375-396.