

# Equidistribution of Hecke Eigenvalues

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# Presentation Overview

- 1 Setup and Notation
- 2 Extending Serre's equidistribution result to  $S_k^{\text{new}}(N, \chi)$
- 3 A Surprise!

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① Setup and Notation

② Extending Serre's equidistribution result to  $S_k^{\text{new}}(N, \chi)$

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  - The newspace  $S_k^{\text{new}}(N)$ .
- **But what about  $S_k^{\text{new}}(N, \chi)$ ???**

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③ A Surprise!

## Step #1: Reduce to Chebyshev polynomials

We want to show that the eigenvalues of  $\widehat{T}_p^{\text{new}}(N, k, \chi)$  are  $d\mu_p$  equidistributed over  $[-2, 2]$  as  $N + k \rightarrow \infty$ . I.e. for all  $g \in C[-2, 2]$ ,

$$\frac{1}{\dim S_k^{\text{new}}(N, \chi)} \sum_{\lambda \in \text{eigv } \widehat{T}_p(N, k, \chi)} g(\lambda) \rightarrow \int_{-2}^2 g(x) d\mu_p(x) \quad \text{as } N + k \rightarrow \infty.$$

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## Lemma 1

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Let  $X_n(x) := U_n(x/2)$  denote the Chebyshev polynomials over  $[-2, 2]$ .

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$$\frac{1}{\dim S_k^{\text{new}}(N, \chi)} \sum_{\lambda \in \text{eigv } \widehat{T}_p(N, k, \chi)} X_n(\lambda) \longrightarrow \int_{-2}^2 X_n(x) d\mu_p(x). \quad (*)$$

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### Lemma 2

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Using the generating function definition for  $X_k(x)$ , one can calculate that

$$\sum_{k=0}^{\infty} X_k(x) \mathbb{1}_{2|k} p^{-k/2} = \frac{p+1}{\pi} \frac{(1-x^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2-x^2} = \frac{d\mu_p(x)}{d\mu_{\infty}(x)}.$$

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$$\begin{aligned} \text{So } \int_{-2}^2 X_n(x) d\mu_p(x) &= \int_{-2}^2 X_n(x) \sum_{k=0}^{\infty} X_k(x) \mathbb{1}_{2|k} p^{-k/2} d\mu_{\infty}(x) \\ &= \sum_{k=0}^{\infty} \mathbb{1}_{2|k} p^{-k/2} \int_{-2}^2 X_n(x) X_k(x) d\mu_{\infty}(x) = \mathbb{1}_{2|n} p^{-n/2}. \end{aligned}$$

## Step #3: Evaluate the LHS of (\*)

The classical recurrence relation for the  $\widehat{T}_{p^n}$  is identical to the defining recurrence relation for the Chebyshev polynomials:

$$\begin{aligned}\widehat{T}_{p^0} &= \text{Id}, & X_0(x) &= 1, \\ \widehat{T}_{p^1} &= \widehat{T}_p, & X_1(x) &= x, \\ \widehat{T}_{p^{n+1}} &= \widehat{T}_p \widehat{T}_{p^n} - \widehat{T}_{p^{n-1}}, & X_{n+1}(x) &= x X_n(x) - X_{n-1}(x).\end{aligned}$$

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This means that the LHS of (\*) is equal to

$$\begin{aligned}\text{LHS} &= \frac{1}{\dim S_k^{\text{new}}(N, \chi)} \sum_{\lambda \in \text{eigv } \widehat{T}_p(N, k, \chi)} X_n(\lambda) \\ &= \frac{1}{\dim S_k^{\text{new}}(N, \chi)} \sum_{\lambda \in \text{eigv } X_n(\widehat{T}_p(N, k, \chi))} \lambda \\ &= \frac{\text{Tr } \widehat{T}_{p^n}^{\text{new}}(N, k, \chi)}{\text{Tr } \widehat{T}_1^{\text{new}}(N, k, \chi)}.\end{aligned}$$

## Step #4: Trace Estimate

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$$\frac{1}{\dim S_k^{\text{new}}(N, \chi)} \sum_{\lambda \in \text{eigv } \widehat{T}_p(N, k, \chi)} X_n(\lambda) \longrightarrow \int_{-2}^2 X_n(x) d\mu_p(x). \quad (*)$$

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This follows from the following trace estimate, completing the proof.

### Trace estimate

$$\text{Tr } \widehat{T}_m^{\text{new}}(N, k, \chi) = \frac{\mathbb{1}_{m=\square}}{\sqrt{m}} \frac{k-1}{12} \psi_f^{\text{new}}(N) + O(N^{1/2+\varepsilon}),$$

where  $f = f(\chi)$  and  $\psi_f^{\text{new}}(N) \geq \frac{N}{4\omega(N)} \gg N^{1-\varepsilon}$ .

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③ A Surprise!

## Pop Quiz

- This will test your intuition regarding how large various spaces of modular forms are.
- There are four True/False questions.
- Everyone has to participate.

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- Consider integers  $N \geq 1$  and integers  $k \geq 2$  even.  
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It turns out that  $\dim S_k^{\text{new}}(\Gamma_0(N), \chi) = 0$  whenever  $2 \mid f(\chi)$ ,  $2 \parallel N/f(\chi)$ .

## An extra condition

In fact, the trace estimate from earlier actually should be:

### Trace estimate

$$\mathrm{Tr} \widehat{T}_m^{\mathrm{new}}(N, k, \chi) = \frac{\mathbb{1}_{m=\square}}{\sqrt{m}} \frac{k-1}{12} \psi_f^{\mathrm{new}}(N) + O\left(N^{1/2+\varepsilon}\right),$$

where  $f = f(\chi)$ . Moreover,  $\psi_f^{\mathrm{new}}(N) \geq \frac{N}{4^{\omega(N)}} \gg N^{1-\varepsilon}$  whenever it is not the case that  $2 \mid f$ ,  $2 \parallel N/f$ .

This was not known at the time Serre published his original paper. This missing tool was the reason Serre did not extend his result to  $S_k^{\mathrm{new}}(N, \chi)$ .

# An extra condition

In fact, the trace estimate from earlier actually should be:

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## Theorem 3

*Assuming it is not the case that  $2 \mid f(\chi)$ ,  $2 \parallel N/f(\chi)$ ; the eigenvalues of  $\hat{T}_p^{\mathrm{new}}(N, k, \chi)$  are  $d\mu_p$ -equidistributed over  $[-2, 2]$  as  $N + k \rightarrow \infty$ .*

## Trace estimate condition

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## Trace estimate condition

$$\psi_f^{\text{new}}(N) \geq \frac{N}{4\omega(N)} \text{ whenever it is not the case that } 2 \mid f, 2 \parallel N/f.$$

One can calculate that  $\psi_f^{\text{new}}(N) = \psi(f) \cdot \beta * \psi_f(N/f)$ , where  $\psi$  and  $\beta * \psi_f$  are the multiplicative functions

$$\psi(p^r) = p^r \left( 1 + \frac{1}{p} \right),$$
$$\beta * \psi_f(p^r) = \begin{cases} p^r \left( 1 - \frac{1}{p} - \left[ \frac{1}{p^2} \right]_{r \geq 2} + \left[ \frac{1}{p^3} \right]_{r \geq 3} \right) & \text{if } p \nmid f, \\ p^r \left( 1 - \frac{2}{p} + \left[ \frac{1}{p^2} \right]_{r \geq 2} \right) & \text{if } p \mid f. \end{cases}$$

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And  $\left( 1 - \frac{1}{p} - \left[ \frac{1}{p^2} \right]_{r \geq 2} + \left[ \frac{1}{p^3} \right]_{r \geq 3} \right) \geq \frac{1}{4}$  for all prime powers  $p^r$ ,

$\left( 1 - \frac{2}{p} + \left[ \frac{1}{p^2} \right]_{r \geq 2} \right) \geq \frac{1}{4}$  for all prime powers  $p^r \neq 2$ .

# The End



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