Stark units, quantum modular cocycles, and complex equiangular lines

Gene S. Kopp

Louisiana State University

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L-functions at $s = 1$: example with base field $\mathbb{Q}$

The following formula can be proved using calculus. Try it!

Example (Exercise)

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} + - - + \cdots = \frac{1}{\sqrt{2}} \log\left(1 + \sqrt{2}\right)$$

- The left-hand side is the value $L(1, \chi)$, where $\chi(n) = \left(\frac{2}{n}\right)$ is the Dirichlet character associated to the field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$.
- On the right-hand side is $1 + \sqrt{2}$, the fundamental unit of $\mathbb{Q}(\sqrt{2})$.
- This talk concerns higher analogs of this formula in the framework of the Stark conjectures. Within this framework, $1 + \sqrt{2}$ is called a Stark unit.

L-functions at $s = 1$: real quadratic example

The following formula is an open conjecture!

Example

$$\sum_{m=1}^{\infty} \sum_{\substack{n \in \mathbb{Z} \\ -\frac{5}{3}m \le n < \frac{5}{3}m}} \frac{e^{-2\pi i m/5} - e^{2\pi i m/5}}{3m^2 - n^2} = \frac{\pi}{i\sqrt{3}} \log\left(\varepsilon\right),$$

where $\varepsilon \approx 3.890861714$ is a root of the polynomial equation

$$\begin{aligned} x^8 &- (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 \\ &+ (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 \\ &- (8+5\sqrt{3})x + 1 = 0. \end{aligned}$$

The number ε is an algebraic unit in a particular cyclic order 8 extension $H_{5\infty_2}$ of $\mathbb{Q}(\sqrt{3})$. This conjecture is part of the Stark conjectures, and the number ε is a Stark unit.

q-Pochhammer asymptotics near real quadratic points: example

For $q = e^{2\pi i\tau}$, set $\varpi_{\binom{0}{4/5}}(\tau) = (q^{4/5}, q)_\infty = \prod_{k=0}^{\infty}(1 - q^{k+4/5})$.

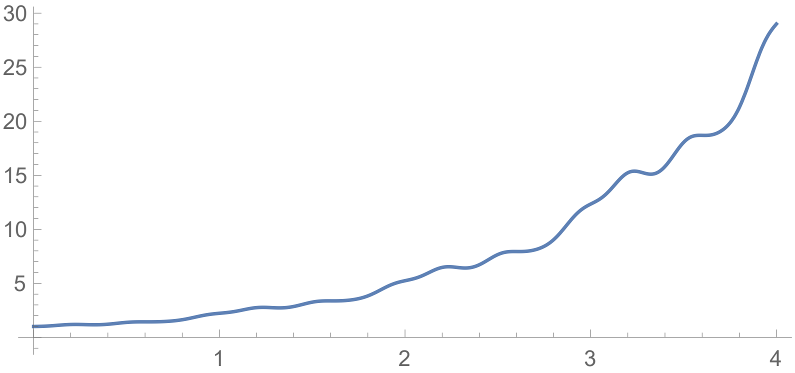

Figure: Graph of $y = \left|\varpi_{\binom{0}{4/5}}\left(\sqrt{3} + i\, e^{-6\log(2+\sqrt{3})t}\right)\right|$

q-Pochhammer asymptotics near real quadratic points: example

For $q = e^{2\pi i \tau}$, set $\varpi_{\binom{0}{4/5}}(\tau) = (q^{4/5}, q)_\infty = \prod_{k=0}^{\infty}(1 - q^{k+4/5})$.

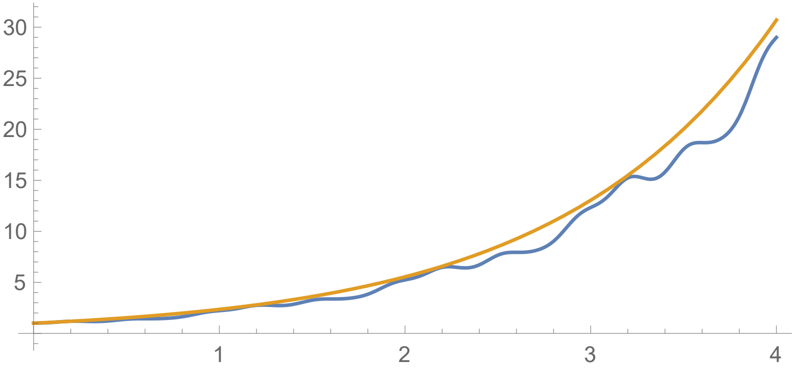

Figure: Graphs of $y = \left|\varpi_{\binom{0}{4/5}}\left(\sqrt{3} + i\, e^{-6\log(2+\sqrt{3})t}\right)\right|$ and $y = (2.35385)^t$

q-Pochhammer asymptotics near real quadratic points: example

For $q = e^{2\pi i\tau}$, set $\varpi_{\binom{0}{4/5}}(\tau) = (q^{4/5}, q)_\infty = \prod_{k=0}^{\infty}(1 - q^{k+4/5})$.

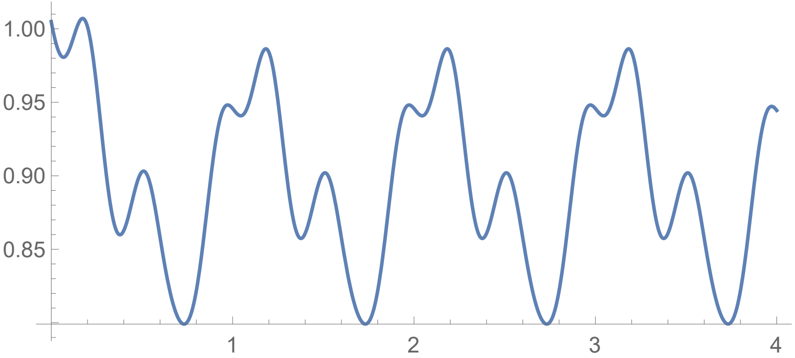

Figure: Graph of $y = \left|\nu^{-t}\varpi_{\binom{0}{4/5}}\left(\sqrt{3} + i\, e^{-6\log(2+\sqrt{3})t}\right)\right|$; $\nu \approx 2.35385\, e^{-\frac{7\pi i}{20}}$.

q-Pochhammer asymptotics near real quadratic points: example

For $q = e^{2\pi i \tau}$, set $\varpi_{\left(\begin{smallmatrix}0\\4/5\end{smallmatrix}\right)}(\tau) = (q^{4/5}, q)_\infty = \prod_{k=0}^{\infty}(1 - q^{k+4/5})$.

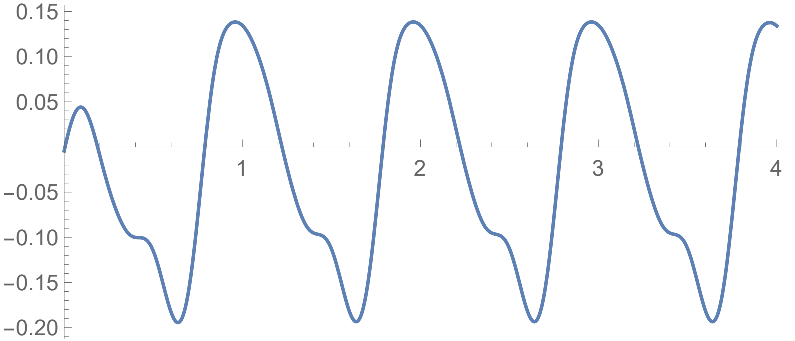

Figure: Graph of $y = \arg\left(\nu^{-t}\varpi_{\left(\begin{smallmatrix}0\\4/5\end{smallmatrix}\right)}\left(\sqrt{3} + i\, e^{-6\log(2+\sqrt{3})t}\right)\right)$.

Interpreting the base of the exponential

What is the meaning of the number $\nu \approx 2.35385\, e^{-\frac{7\pi i}{20}}$?

The phase $e^{-\frac{7\pi i}{20}}$ can be obtained up to ± 1 from the modularity of $(q^{1/5}, q)_\infty (q^{4/5}, q)_\infty$.

The number $|\nu|^2 \approx 5.54061$ appears to be a root of the polynomial equation

$$\begin{aligned} x^8 &- (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 \\ &+ (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 \\ &- (8+5\sqrt{3})x + 1 = 0. \end{aligned}$$

Same polynomial as in Example 1! The root $|\nu|^2$ is also a Stark unit (a Galois conjugate of the Stark unit in Example 1).

The number $|\nu| \approx 2.35385$ appears to be an element of $H_{5\infty_2}(\sqrt{2})$.

SIC-POVMs

Definition

A SIC or SIC-POVM (symmetric, informationally complete, positive operator-valued measure) is (an set of quantum measurements equivalent) to d^2 equiangular lines in $\mathbb{C}^d$.

Example (SIC for $d = 2$)

$$\left\{ \left[1 : \frac{1+i}{1+\sqrt{3}}\right], \left[1 : \frac{-1-i}{1+\sqrt{3}}\right], \left[\frac{1+i}{1+\sqrt{3}} : 1\right], \left[\frac{-1-i}{1+\sqrt{3}} : 1\right] \right\} \subset \mathbb{P}^1(\mathbb{C})$$

Conjecture (Zauner 1999)

There is at least one SIC in every dimension $d \geq 1$.

Zauner's Conjecture has been verified for (at least) $d \leq 53$.

Example ($d = 5$)

The unique SIC in dimension 5 is given by equiangular lines

$$\{\mathbb{C}\mathbf{v}_1, \mathbb{C}\mathbf{v}_2, \ldots, \mathbb{C}\mathbf{v}_{25}\},$$

for unit vectors $\mathbf{v}_i$ whose inner products for $i \neq j$ are

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \frac{1}{\sqrt{6}} \eta_{ij},$$

where $|\eta_{ij}| = 1$, and the η_{ij}^2 are (up to well-understood fifth roots of unity) the roots of the polynomial

$$\begin{aligned} x^8 &- (8 - 5\sqrt{3})x^7 + (53 - 30\sqrt{3})x^6 - (156 - 90\sqrt{3})x^5 \\ &+ (225 - 130\sqrt{3})x^4 - (156 - 90\sqrt{3})x^3 + (53 - 30\sqrt{3})x^2 \\ &- (8 - 5\sqrt{3})x + 1 = 0, \end{aligned}$$

that is, Galois conjugates (over $\mathbb{Q}$) of (putative) Stark units!

Explicit class field theory and Hilbert's 12th Problem

Problem

Given a number field F, describe a cofinal set of abelian extensions of F that are practically computable, arise naturally as special values of analytic functions, and/or arise naturally from algebraic geometry (preferably all three!).

- For $F = \mathbb{Q}$, consider cyclotomic fields $\mathbb{Q}(\zeta_n)$ (Kronecker–Weber theorem).
- For F imaginary quadraic, consider division fields $F(E[n])$ of an elliptic curve with CM by an order of F (Kronecker's Jugentraum, theory of singular moduli).
- For F real quadratic (or at least $F = \mathbb{Q}(\sqrt{3})$), examples 1 and 2 point toward a possible analytic solution, and example 3 points toward a possible algebro-geometric solution.

Multiplicative holomorphic quantum modular forms

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.

Definition (MHQMF)

A multiplicative holomorphic quantum modular form of level Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ whose transformation factor

$$w_A(\tau) = \frac{f\left(\frac{a\tau+b}{c\tau+d}\right)}{f(\tau)} \qquad \text{where } A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$$

analytically continues to $\mathbb{C}_A = \{\tau \in \mathbb{C} : c\tau + d \notin (-\infty, 0]\}$.

The transformation factor is also called a cocycle because it satisfies the 1-cocycle relation

$$w_{AB}(\tau) = (w|_0 A)(B \cdot \tau) w_B(\tau).$$

We call w_A a coboundary if there exists a corresponding f that itself analytically continues to $\mathbb{C}_A$.

The q-Pochhammer symbol as a MHQMF

For $\mathbf{r} \in \binom{r_1}{r_2} \in \mathbb{Q}^2$, $\tau \in \mathbb{H}$, and $\mathrm{e}(z) := e^{2\pi i z}$, consider the function

$$\varpi_{\mathbf{r}}(\tau) = (\mathrm{e}(r_2\tau - r_1), \mathrm{e}(\tau))_\infty = \prod_{k=0}^{\infty} (1 - \mathrm{e}((k + r_2)\tau - r_1)).$$

Let $\Gamma_{\mathbf{r}} = \{A \in \mathrm{SL}_2(\mathbb{Z}) : A\mathbf{r} - \mathbf{r} \in \mathbb{Z}^2\}$.

Theorem (consequence of Shintani 1977; K 2024)

The function $\varpi_{\mathbf{r}}(\tau)$ is a multiplicative holomorphic quantum modular form of level $\Gamma_{\mathbf{r}}$. That is, for each $A \in \Gamma_{\mathbf{r}}$, there is a holomorphic function $\mathfrak{w}^{\mathbf{r}}_A(\tau)$ on $\mathbb{C}_A$ with the property that

$$\varpi_{\mathbf{r}}(A \cdot \tau) = \mathfrak{w}^{\mathbf{r}}_A(\tau)\varpi_{\mathbf{r}}(\tau).$$

We call $\mathfrak{w}^{\mathbf{r}}_A(\tau)$ the shin function and $A \mapsto \mathfrak{w}^{\mathbf{r}}_A$ the Shintani–Faddeev cocycle. (Up to change of variables, $\mathfrak{w}^{\mathbf{r}}_S$ is the **Shintani double sine function** or the **Faddeev quantum dilogarithm**.)

Real multiplication (RM) values of a modular cocycle

- Let Γ be a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.
- Let $w = (w_A)_{A \in \Gamma}$ be a multiplicative quantum modular cocycle.
- Consider a real quadratic number β.
- Let $A_\beta^+ \in \Gamma$ be the "positive" generator for the stabilizer of β in Γ (or in $\Gamma/\{\pm I\}$ if $-I \in \Gamma$), with $A_\beta^+ \cdot \binom{\beta}{1} = \lambda \binom{\beta}{1}$ for $\lambda > 1$.
- If $\beta \in \mathbb{C}_A$, call the value $w[\beta] := w_{A_\beta^+}(\beta)$ the real multiplication (RM) value of w at β.
- Then $w[\beta]$ depends only on β and the cohomology class of w.
- Compare and contrast to p-adic rigid meromorphic cocycles of Darmon and Vonk (2021), Darmon, Pozzi, and Vonk (2021).

RM values of the standard weight cocycle

Note $j_A(\tau) := c\tau + d$ defines a modular cocycle for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Proposition (Exercise)

Let β be a real quadratic irrational and $\mathfrak{b} = \beta\mathbb{Z} + \mathbb{Z}$. Let

$$\mathcal{O} = (\mathfrak{b} : \mathfrak{b}) = \{\alpha \in \mathcal{O} : \alpha\mathfrak{b} \subseteq \mathfrak{b}\}$$

be the multiplier ring of $\mathfrak{b}$ (an order in $\mathbb{Q}(\beta)$). Then

$$j[\beta] = \varepsilon_{\mathcal{O}}^{+} \in \mathcal{O}^{\times}$$

where $\varepsilon_{\mathcal{O}}^{+} > 1$ generates the totally positive part of $\mathcal{O}^{\times}$.

Main theorem: RM values of the Shintani–Faddeev modular cocycle

Theorem (K 2024, special case)

- Let F be an real quadratic number field.
- Let $\mathfrak{m} \subseteq \mathcal{O}_F$ be a nonzero ideal with $\mathfrak{m} \neq \mathcal{O}_F$.
- Let $\mathfrak{A} \in \mathrm{Cl}_{\mathfrak{m}\infty_2}$ and $\mathfrak{A}_0$ the class of $\mathfrak{A}$ in $\mathrm{Cl}_{\mathcal{O}_F}$.
- Choose $\mathfrak{b} \in \mathfrak{A}_0^{-1}$ coprime to $\mathfrak{m}$.
- Write $\mathfrak{bm} = \alpha(\beta\mathbb{Z} + \mathbb{Z})$, α totally positive, $\beta > \beta'$.
- Choose $\mathbf{r} = \binom{r_1}{r_2} \in \mathbb{Q}^2$ such that $(\alpha(r_2\beta - r_1))\mathfrak{b}^{-1} \in \mathfrak{A}$ and $r_2\beta' - r_1 > 0$.
- Suppose that $\mathcal{O}_F^\times \cap (1 + \mathfrak{m})$ consists of only totally positive units.

Then, for certain characters ψ^2 and $\chi_\mathbf{r}$ on $\Gamma_\mathbf{r}$, which we treat as constant cocycles,

$$\exp\left(-2\zeta'_{\mathfrak{m}\infty_2}(0, \mathfrak{A})\right) = (\psi^2\,\chi_\mathbf{r}\,(\text{ש}^\mathbf{r})^{-2})[\beta].$$

RM values of the Shintani–Faddeev modular cocycle: an example

The following special value was computed to high precision and repeats an example from earlier in the talk. Equality is conditional on the Stark conjectures.

$$\text{ש}^{\left(\substack{0\\4/5}\right)}\left[\sqrt{3}\right] = \text{ש}^{\left(\substack{0\\4/5}\right)}_{\left(\substack{26\ 45\\15\ 26}\right)}\left(\sqrt{3}\right) \stackrel{?}{=} e^{-7\pi i/20}\sqrt{u}$$

where $u \approx 5.54061$ is a root of the polynomial equation

$$\begin{aligned} x^8 &- (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 \\ &+ (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 \\ &- (8+5\sqrt{3})x + 1 = 0. \end{aligned}$$

The Twisted Convolution Conjecture

Conjecture (Appleby, Flammia, and Kopp 2025)

Fix positive integers (d, n, r) such that $nr(d-r) = d^2 - 1$ and $0 < r < \frac{d-1}{2}$. Let $\Delta = n(n-4) = f^2\Delta_0$.

Let Q be an integral binary quadratic form of $\mathrm{disc}(Q) = (f')^2\Delta_0$ with $f' \mid f$.

Fix β such that $Q(\beta, 1) = 0$, let $A = A_\beta^+$ be the positive stabilizer of β in $\Gamma(d)$, and write $A = L^{2m+1}$ (where m is determined by the other parameters with details omitted).

If $\mathbf{p} \in \mathbb{Z}^2/d\mathbb{Z}^2$ such that $\mathbf{p} \neq \mathbf{0}$, then

$$\sum_{\mathbf{q}\in\mathbb{Z}^2/d\mathbb{Z}^2} \zeta_d^{r\,\mathbf{p}^\top SL\mathbf{q}}\, \varpi_A^{d^{-1}\mathbf{q}}(\beta)\, \varpi_{A^{-1}}^{d^{-1}(\mathbf{q}-\mathbf{p})}(\beta) = 0.$$

Theorem (Appleby, Flammia, and Kopp 2025)

Assume the Twisted Convolution Conjecture and the rank 1 real quadratic Stark conjecture. Then, Zauner's conjecture is true (SICs exist in every dimension d.)

In fact, under these conjectures, one can say much more:

- There is an explicit construction of SICs under which each geometric equivalence class of SIC is labeled by a class of binary quadratic forms.
- One also obtains subspace generalizations called r-SICs (or maximal equichordal fusion frames).
- If F is a real quadratic field whose fundamental unit has odd trace, then every abelian extension of F is a subfield of an abelian extension of F generated by the invariants of an r-SIC.

Thank you!

Thank you to the organizers!

Thank you for listening! Any questions?

G. S. Kopp. The Shintani–Faddeev modular cocycle: Stark units from q-Pochhammer ratios, arXiv:2411.06763. Preprint, 2024.

M. Appleby, G. S. Kopp, and S. T. Flammia. A constructive approach to Zauner's conjecture via the Stark conjectures, arXiv:2501.03970. Preprint, 2025.