

Automorphic Forms and Automorphic Representations

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The Rational Numbers

On \mathbb{Q} we have...

- the usual (archimedean) absolute value $|\cdot|_\infty$
- the p -adic absolute value $|\cdot|_p$, for each prime p

Theorem (Ostrowski)

Up to equivalence, these are the only (non-trivial) absolute values on \mathbb{Q} .

Completion (“Cauchy sequences modulo null sequences”) of \mathbb{Q} with respect to...

- $|\cdot|_\infty$ gives the field of real numbers $\mathbb{R} = \mathbb{Q}_\infty$
- $|\cdot|_p$ gives the field of p -adic numbers \mathbb{Q}_p

In \mathbb{Q}_p we have the ring of **p -adic integers** \mathbb{Z}_p and its unit group \mathbb{Z}_p^\times .

- The ring of **adeles** is the **restricted direct product**

$$\mathbb{A} = \left\{ (x_p)_{p \leq \infty} \in \prod_{p \leq \infty} \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for almost all } p \right\}.$$

Hence $\mathbb{A} \subset \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \dots$

- The group of **ideles** is the **restricted direct product**

$$\mathbb{A}^\times = \left\{ (x_p)_{p \leq \infty} \in \prod_{p \leq \infty} \mathbb{Q}_p^\times \mid x_p \in \mathbb{Z}_p^\times \text{ for almost all } p \right\}.$$

Hence $\mathbb{A}^\times \subset \mathbb{R}^\times \times \mathbb{Q}_2^\times \times \mathbb{Q}_3^\times \times \mathbb{Q}_5^\times \times \dots$

Idea: Look at all completions at the same time.

\mathbb{Q}_p carries the topology induced by $|\cdot|_p$.

- $\mathbb{Q}_p, \mathbb{Q}_p^\times$ are **locally compact topological groups**.

\mathbb{A} and \mathbb{A}^\times carry the **restricted direct product topology**.

- $\mathbb{A}, \mathbb{A}^\times$ are also **locally compact topological groups**.

Hence we have the tools of **harmonic analysis**: a **Haar measure**, a theory of **integration**, a **Fourier transform**, **Pontryagin duality**, . . .

These are critical tools in the theory of automorphic forms.

Adelized Algebraic Groups

Adelization of $G = \mathrm{GL}(2)$:

$$G(\mathbb{A}) = \left\{ (g_p)_{p \leq \infty} \in \prod_{p \leq \infty} G(\mathbb{Q}_p) \mid g_p \in G(\mathbb{Z}_p) \text{ for almost all } p \right\}.$$

Hence $\mathrm{GL}(2, \mathbb{A}) \subset \mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{Q}_2) \times \mathrm{GL}(2, \mathbb{Q}_3) \times \mathrm{GL}(2, \mathbb{Q}_5) \times \dots$

$G(\mathbb{A})$ is endowed with the **restricted direct product topology**, making it a **locally compact topological group**. Again we have a Haar measure and integration. **Harmonic analysis gets more complicated, because $G(\mathbb{A})$ is no longer abelian.** (But we have many abelian subgroups!)

In the same way we can **adelize** any algebraic \mathbb{Q} -group, for example $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{SO}(n)$, $\mathrm{Sp}(2n)$.

Example

$$\mathbb{A}^\times = \mathrm{GL}(1, \mathbb{A})$$

Two Famous Works

The first time “mathematical principles” (Calculus) are systematically applied to “natural philosophy” (Physics):

Isaac Newton, 1687

“Philosophiæ Naturalis Principia Mathematica”

The first time Fourier analysis is systematically applied to number theory:

John Tate, 1950

“Fourier Analysis in Number Fields and Hecke’s Zeta-Functions”

Rational Points

We have a diagonal embedding

$$\mathbb{Q} \hookrightarrow \mathbb{A}, \quad x \mapsto (x, x, x, \dots)$$

In this way \mathbb{Q} becomes a **discrete** and **co-compact** subgroup of \mathbb{A} .
(Analogous to $\mathbb{Z} \hookrightarrow \mathbb{R}$.)

Similarly we have $\mathbb{Q}^\times \hookrightarrow \mathbb{A}^\times$, still **discrete**, but **not co-compact**.

Important Fact

$$\mathbb{A}^\times = \mathbb{Q}^\times \times \mathbb{R}_{>0} \times \prod_{p < \infty} \mathbb{Z}_p^\times$$

For $G = \mathrm{GL}(n)$ and many other groups we have similarly:

Strong Approximation

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+ \prod_{p < \infty} G(\mathbb{Z}_p) \quad (\text{But not direct!})$$

Automorphic Forms

Let G be an algebraic \mathbb{Q} -group. An **automorphic form** on $G(\mathbb{A})$ is a function $\Phi : G(\mathbb{A}) \rightarrow \mathbb{C}$ with the following properties:

- 1 Φ is **left-invariant under rational points**, i.e.

$$\Phi(\gamma g) = \Phi(g) \quad \text{for all } \gamma \in G(\mathbb{Q}).$$

(This is the quintessential automorphic property.)

- 2 Φ is **smooth**. (Locally constant as a function of the p -adic places.)
- 3 Φ is **K -finite**. (True if Φ has a **weight** as a function of $G(\mathbb{R})$.)
- 4 Φ is **\mathcal{Z} -finite**. (“ Φ satisfies a differential equation.”)
- 5 Φ is of **moderate growth**. (Similar to “holomorphy at the cusps.”)

Why Automorphic Forms?

Philosophy

- There is a very successful theory of \mathbb{Z} -periodic functions on \mathbb{R} (classical Fourier analysis).
- We hope to get something similarly successful, with a number-theoretic flavor, for $G(\mathbb{Q})$ -periodic functions on $G(\mathbb{A})$.

But then...

Why not just look at $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$, instead of the above regularity conditions? After all, classical Fourier analysis is done on $L^2(\mathbb{Z} \backslash \mathbb{R})$.

People do that! However...

- Dealing with L^2 -spaces requires functional analysis. (too hard)
- Automorphic forms lead to a nice algebraic theory.
- Many interesting automorphic forms are not in L^2 .

The Space of Automorphic Forms

Let $\mathcal{A}(G)$ be the space of automorphic forms on $G(\mathbb{A})$. Observations:

- If $\Phi \in \mathcal{A}(G)$ and $h \in G(\mathbb{A}_{\text{fin}}) = \prod'_{p < \infty} G(\mathbb{Q}_p)$, then the function

$$g \mapsto \Phi(gh) \quad (\text{right translation})$$

is also in $\mathcal{A}(G)$.

- Not quite true for $h \in G(\mathbb{R})$ (because of K -finiteness), but still true for $h \in K$.
- If $X \in \mathfrak{g}$ (the **real Lie algebra**), then the function $X.\Phi$, defined by

$$g \mapsto \left. \frac{d}{dt} \right|_0 \Phi(g \exp(tX)),$$

is also in $\mathcal{A}(G)$.

Hence: $\mathcal{A}(G)$ is simultaneously a $G(\mathbb{A}_{\text{fin}})$ -module and a (\mathfrak{g}, K) -module.

Common abuse of terminology

“ $G(\mathbb{A})$ acts on $\mathcal{A}(G)$ ”, or “ $\mathcal{A}(G)$ is a representation of $G(\mathbb{A})$.”

Definition

A representation (π, V) of $G(\mathbb{A})$ is called **automorphic**, if it is isomorphic to an **irreducible subquotient** of $\mathcal{A}(G)$.

- In most cases we will have $V \subset \mathcal{A}(G)$.
- But in general we only have

$$0 \subset W_2 \subset W_1 \subset \mathcal{A}(G)$$

with $V \cong W_1/W_2$ as $G(\mathbb{A})$ -representations.

The Tensor Product Theorem

Suppose we have **local representations** as follows:

- An irreducible, **admissible** (\mathfrak{g}, K) -module (π_∞, V_∞) .
- For each finite prime an irreducible **admissible** representation (π_p, V_p) of $G(\mathbb{Q}_p)$.
- For almost all primes π_p is **spherical**, i.e., it admits a non-zero $G(\mathbb{Z}_p)$ -invariant vector.

Then we can form the **restricted tensor product** $V = \otimes V_p$, which carries an irreducible, admissible representation $\pi = \otimes \pi_p$ of $G(\mathbb{A})$.

Tensor Product Theorem (Flath, 1979)

Let (π, V) be an irreducible, admissible representation of $G(\mathbb{A})$. Then there exist local representations as above such that $\pi \cong \otimes \pi_p$.

Note: This has nothing to do with automorphic representations (but it applies to automorphic representations).

Example: Induced Representations on $GL(2)$

Fix an even integer $k \geq 4$. Consider functions $f : GL(2, \mathbb{Q}_p) \rightarrow \mathbb{C}$ with the transformation property

$$f\left(\begin{bmatrix} a & b \\ & d \end{bmatrix} g\right) = \left|\frac{a}{d}\right|_p^{k/2} f(g).$$

- For $\underline{p = \infty}$, let V_∞ be the space of all such functions that also have a **weight** ℓ , i.e.,

$$f(gr(\theta)) = e^{i\ell\theta} f(g) \quad \text{for } r(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$

Then V_∞ is a (\mathfrak{g}, K) -module under right translation.

- For $\underline{p < \infty}$, let V_p be the space of all **locally constant** such functions. Then V_p is a $G(\mathbb{Q}_p)$ -representation under right translation.

Hence we have **local representations** (π_p, V_p) for each p . Let $V = \otimes V_p$ and $\pi = \otimes \pi_p$. Then (π, V) is a **global induced representation**.

Distinguished Vectors

Now choose the following f_p 's in the local induced representations:

- For $\underline{p} = \infty$, let f_∞ be the **normalized weight k -vector**, i.e.,

$$f_\infty\left(\begin{bmatrix} a & b \\ & d \end{bmatrix} r(\theta)\right) = \left|\frac{a}{d}\right|_\infty^{k/2} e^{ik\theta}. \quad (\text{well-defined, because } k \text{ is even})$$

- For $\underline{p} < \infty$, let f_p be the **normalized spherical vector**, i.e.,

$$f_p\left(\begin{bmatrix} a & b \\ & d \end{bmatrix} \kappa\right) = \left|\frac{a}{d}\right|_p^{k/2} \quad \text{for } \kappa \in \text{GL}(2, \mathbb{Z}_p).$$

Then $f = \otimes f_p$ is a **globally distinguished vector**. It is **not an automorphic form**, but it is halfway there:

$$f\left(\begin{bmatrix} a & b \\ & d \end{bmatrix} g\right) = f(g) \quad \text{for } a, d \in \mathbb{Q}^\times, b \in \mathbb{Q}.$$

In other words, $f(\gamma g) = f(g)$ for $\gamma \in B(\mathbb{Q})$, where B is the upper triangular subgroup of $\text{GL}(2)$.

Adelic Eisenstein Series

We now force the rest of the automorphic transformation property:

$$\mathbb{E}(g) := \sum_{\gamma \in B(\mathbb{Q}) \backslash \mathrm{GL}(2, \mathbb{Q})} f(\gamma g). \quad (\text{ignore convergence for now})$$

Properties:

- \mathbb{E} is left-invariant under $\mathrm{GL}(2, \mathbb{Q})$. In fact, \mathbb{E} is an automorphic form!
- \mathbb{E} is right-invariant under $\prod_{p < \infty} \mathrm{GL}(2, \mathbb{Z}_p)$.

Recall Strong Approximation

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+ \prod_{p < \infty} G(\mathbb{Z}_p).$$

Hence \mathbb{E} is determined by its values on $\mathrm{GL}(2, \mathbb{R})^+$. Moreover:

- \mathbb{E} is invariant under scalar matrices (because f has this property).

Hence \mathbb{E} is determined by its values on $\mathrm{SL}(2, \mathbb{R})$.

Descent to the Upper Half Plane

The Upper Half Plane is a Homogeneous Space for $SL(2, \mathbb{R})$

$$\mathcal{H} = SL(2, \mathbb{R})/SO(2) \text{ via } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{ai + b}{ci + d}.$$

Suppose ϕ is any function on $SL(2, \mathbb{R})$ with the **weight k property**:

$$\phi(gr(\theta)) = e^{ik\theta} \phi(g).$$

Define a function $F : \mathcal{H} \rightarrow \mathbb{C}$ by

$$(F|_k g)(i) = \phi(g)$$

i.e.,

$$F(\tau) = (ci + d)^k \phi(g), \quad \text{where } g \in SL(2, \mathbb{R}) \text{ is such that } gi = \tau.$$

Even more concretely,

$$F(x + iy) = y^{-k/2} \phi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{bmatrix}\right).$$

Descent of the Eisenstein Series

$$\begin{aligned} F(\tau) &= y^{-k/2} \mathbb{E} \left(\underbrace{\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{bmatrix}}_{\text{at } \infty \text{ only}} \right) \\ &= y^{-k/2} \sum_{\gamma \in B(\mathbb{Q}) \backslash GL(2, \mathbb{Q})} f \left(\gamma \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{bmatrix} \right) \\ &= y^{-k/2} \sum_{\gamma \in \Gamma_\infty \backslash SL(2, \mathbb{Z})} f \left(\gamma \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{bmatrix} \right) \quad (\Gamma_\infty = \begin{bmatrix} \pm 1 & \mathbb{Z} \\ & \pm 1 \end{bmatrix}) \\ &= y^{-k/2} \sum_{\gamma \in \Gamma_\infty \backslash SL(2, \mathbb{Z})} f_\infty \left(\gamma \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{bmatrix} \right) \prod_{p < \infty} f_p(\gamma) \\ &= y^{-k/2} \sum_{\gamma \in \Gamma_\infty \backslash SL(2, \mathbb{Z})} f_\infty \left(\gamma \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{bmatrix} \right). \quad (f_p \text{ spherical}) \end{aligned}$$

The adeles have disappeared at this point.

Continuing the Calculation

$$F(\tau) = y^{-k/2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2, \mathbb{Z})} f_\infty \left(\underbrace{\gamma \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{bmatrix}}_{\text{write this in Iwasawa form}} \right)$$

$$= y^{-k/2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2, \mathbb{Z})} f_\infty \left(\underbrace{\begin{bmatrix} 1 & x' \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y'} & \\ & \sqrt{y'}^{-1} \end{bmatrix} r(\theta)}_{\text{with some } x', y', \theta} \right)$$

$$= y^{-k/2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2, \mathbb{Z})} (y')^{k/2} e^{ik\theta} \quad (\text{by definition of } f_\infty)$$

$$= y^{-k/2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2, \mathbb{Z})} \left(\frac{y}{|c\tau + d|^2} \right)^{k/2} \left(\frac{c\bar{\tau} + d}{|c\tau + d|} \right)^k \quad \left(\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

$$= \dots$$

Result: \mathbb{E} descends to E_k

$$F(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2, \mathbb{Z})} \frac{1}{(c\tau + d)^k}$$

(**Now** we know convergence.)

Summary of the Calculation

The adelic Eisenstein series \mathbb{E} descends to the classical holomorphic weight k Eisenstein series E_k on the upper half plane.

General Descent Procedure for $G = \mathrm{GL}(2)$

- Let (π, V) be an automorphic representation of $G(\mathbb{A})$.
- Assume $V \subset \mathcal{A}(G)$.
- We know $\pi \cong \otimes \pi_p$ with local representations (π_p, V_p) . **Important:** This is an **abstract** isomorphism. The V_p are **some models** for the local representations.
- Choose a **distinguished vector** v_p in each V_p .
(For $p < \infty$: Theory of **local newforms**.)
- Via $V \cong \otimes V_p$, the pure tensor $\otimes v_p$ corresponds to an automorphic form Φ .
- If the v_p are chosen wisely, then, by strong approximation, Φ is determined by its values on $\mathrm{SL}(2, \mathbb{R})$.
- Descend to a function F on \mathcal{H} . It will be a classical modular form of some kind. **In fact, it will be a Hecke eigenform.**

Principle

All classical modular forms arise in this way.

- Start with $F \in S_k(\Gamma_0(N))$.
- Define a function Φ on $SL(2, \mathbb{R})$ via $\Phi(g) = (F|_k g)(i)$.
- Extend Φ trivially to $GL(2, \mathbb{R})^+$.
- By strong approximation, Φ extends to a function on $GL(2, \mathbb{A})$.
- Φ is a **cuspidal automorphic form**.
- Let V be the space spanned by all $(X\Phi)(\cdot h)$, where $X \in \mathfrak{g}$ and $h \in GL(2, \mathbb{A}_{\text{fin}})$.

Theorem

If Φ is an eigenform, then V is an **irreducible** representation of $GL(2, \mathbb{A})$.

Hence in this case we have an **automorphic representation** (π, V) .

The Correspondence Theorem

Theorem

There is a 1-1 correspondence

Cuspidal automorphic representations
 $\pi \cong \otimes \pi_p$ of $GL(2, \mathbb{A})$ with
 $\pi_\infty =$ holomorphic discrete series
representation of lowest weight k



Eigen-newforms
in $S_k(\Gamma_0(N))$

$$\left\{ \begin{array}{l} (\pi, V) \text{ with} \\ V \subset \mathcal{A}(G) \\ \text{and } \pi \cong \otimes \pi_p \end{array} \right\} \mapsto \left\{ \begin{array}{l} \text{distinguished} \\ \text{vector } v_p \text{ in} \\ \text{each } \pi_p \end{array} \right\} \mapsto \left\{ \begin{array}{l} \otimes v_p \in \otimes V_p \\ \text{corresponds} \\ \text{to } \Phi \in V \end{array} \right\} \mapsto \left\{ \begin{array}{l} \Phi \text{ descends} \\ \text{to newform} \\ F \in S_k(\Gamma_0(N)) \end{array} \right\}$$

- Bad choice of v_p for $p < \infty$: F will be an **oldform**.
- Bad choice of v_∞ : F will be **nearly holomorphic**.

Factorization of Modular Forms

$$\begin{array}{ccc} \Phi \in \mathcal{A}(G) & \xrightarrow{\text{generates}} & \left\{ V \subset \mathcal{A}(G) \text{ carrying} \right. \\ & & \left. \begin{array}{l} \text{an irreducible repre-} \\ \text{sentation } \pi \end{array} \right\} & \xrightarrow{\text{factors}} & \pi \cong \otimes \pi_p \\ \uparrow \text{lifts to} & & & & \\ F \in S_k(\Gamma_0(N)) & & & & \end{array}$$

Interesting Question

What are the local representations π_p ?

In this sense we can factor modular forms!

Other Groups, Other Fields

group	field	π_∞	classical objects
GL(1)	\mathbb{Q}	$1_{\mathbb{R}^\times}$	Even Dirichlet characters
GL(1)	\mathbb{Q}	$\text{sgn}_{\mathbb{R}^\times}$	Odd Dirichlet characters
GL(2)	\mathbb{Q}	discrete series	Holomorphic modular forms
GL(2)	\mathbb{Q}	principal series	Maass forms
GL(2)	real	discrete series	Hilbert modular forms
GL(2)	imag. quadr.	discrete series	Bianchi modular forms
GSp(2n)	\mathbb{Q}	discrete series	Siegel modular forms
GSp(2n)	real	discrete series	Hilbert-Siegel modular forms
GU(m, n)	\mathbb{Q}	discrete series	Hermitian modular forms
G_2	\mathbb{Q}	quaternionic d.s.	Quaternionic modular forms
GSpin(15)	$\mathbb{Q}(\sqrt[3]{2})$	whatever	???

Recall the Definition

An **automorphic form** on $G(\mathbb{A})$ is a function $\Phi : G(\mathbb{A}) \rightarrow \mathbb{C}$ s.t.

- 1 Φ is **left-invariant under rational points**
- 2 Φ is **smooth**
- 3 Φ is **K -finite**
- 4 Φ is **\mathcal{Z} -finite**
- 5 Φ is of **moderate growth**

- To obtain **weak modular forms**: Relax the growth condition.
- To obtain **mock modular forms**: Relax K -finiteness.
- To obtain **Jacobi forms**: Allow certain non-reductive groups (Jacobi group $G^J = \mathrm{SL}(2) \ltimes H$)
- To obtain **half-integral weight forms**: Allow certain non-algebraic groups (metaplectic group $\widetilde{\mathrm{SL}}(2)$)

Thank You!